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APPROXIMATION PROPERTIES FROM A MATHEMATICAL-PHYSICAL PERSPECTIVE. POSSIBLE CORRELATIONS WITH THE NEURONAL NETWORK FRACTALITY

BY

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Abstract. In this paper, we consider different problems concerning regularity viewed as an approximation property in several hit-and-miss hypertopologies as a foundation for self-similarity and fractality from a mathematical - physical perspective. Since in some examples of fractals, as the neuronal network or the circulatory system, the uniform property of the Hausdorff hypertopology is not appropriate, the Wijsman hypertopology may be preferred because it could describe better the pointwise properties of fractals.

Keywords: hit-and-miss hypertopologies; Hausdorff topology; Wijsman topology; regularity; approximations; fractal theories; non-differentiable physics; Scale Relativity Theory.

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1. Introduction

Hypertopologies represent an useful tool in optimization, convex analysis, economics, image processing, sound analysis and synthesis and many other fields. In this sense, important theoretical and practical results have been established (Beer, 1993; Apreutesei, 2003; Hu & Papageorgiou, 1997) - concerning Vietoris topology, (di Lorenzo & di Maio, 2006) - melodic similarity, Lu *et al*. (2001) - word image matching, involving Hausdorff-Pompeiu metric. Approaches of topology in psychology have been also obtained (Lewin *et al.*, 1936; Brown, 2012).

All hypertopologies known so far are of the hit-and-miss type, which led to their unification under a single one - the Bombay Hypertopology (di Maio & Naimpally, 2008).

The idea of modeling phenomena behavior at multiple scales has become a useful tool in pure and applied mathematics and physics. Fractal-based techniques lie at the heart of these areas, since fractals are multi-scale objects, which often describe such phenomena better than traditional mathematical models. Hyperspace theories concerning the Hausdorff metric and the Vietoris topology have been developed as a foundation for self-similarity and fractality (Kunze *et al*., 2012; Wicks, 1991).

Topological methods facilitate the study of the dynamical systems chaotic nature (Sharma & Nagar, 2010; Wang *et al*., 2009; Goméz-Rueda *et al*., 2012; Li, 2012; Liu *et al.*, 2009; Ma *et al*., 2009; Fu & Xing, 2012), since they are collective phenomenon emerging out of many segregated components. Most of these systems are collective (set-valued) dynamics of many units of individual systems. The underlying dynamics of the dynamical systems is set-valued, collective (Edalat, 1995; Gavriluț & Agop, 2013).

Since many phenomena with complex patterns and structures are widely observed in brain, this permits the mathematical modeling and analysis of neuronal networks from the dynamical systems set-valued viewpoint. These phenomena are some manifestations of a multidisciplinary paradigm called emergence or complexity. They share a common unifying principle of dynamic arrays. Precisely, interconnections of a sufficiently large number of simple dynamic units can exhibit complex and self-organizing behaviors. System's complexity can be measured by the topological entropy of the topological dynamical system. In this sense, it has been proposed a new definition of the topological entropy for continuous mappings on arbitrary topological spaces (Liu *et al.*, 2009).

In recent years, in computational graphics and automatic recognition of figures problems, it was necessary to measure accurately the matching, *i.e.* to calculate the distance between two sets of points. So, one should use an acceptable distance which has to satisfy the first condition in the definition of a distance - the distance is zero if and only if the overlap is perfect. An appropriate metric is the Hausdorff metric measuring the degree of overlap of two compact sets. In some examples of fractals (the neuronal networks, the circulatory system), Hausdorff topology is not appropriate due to its uniform property. As a more convenient topology on the set of values of the multifunctions which we study, Wijsman topology may be preferred instead of the Hausdorff one, because Wijsman topology could describe better the pointwise properties of fractals. Recently, generalized fractals in hyperspaces endowed with Hausdorff, or with Vietoris hipertopology have been studied (Andres & Fišer, 2004; Andres & Rypka, 2012; Banakh & Novosad, 2013; Kunze *et al.*, 2012).

On the other hand, together with the increasing interest in hypertopologies, non-additive set-valued measure theory continued to develop. In this context, regularity is known as an important continuity property with respect to different topologies. At the same time, it can be viewed as an approximation property of different "unknown" sets by other sets for which we have more information (Gavriluț & Agop, 2015a; Gavriluț & Agop, 2015b). From a mathematical perspective, this approximation is usually done from the left by closed sets (or more restrictive, by compact sets) and/or from the right by open sets. As a theoretical application of regularity, classical Lusin theorem concerns with the existence of continuous restrictions of measurable functions and it is a very important and useful result for discussing different kinds of approximation of measurable functions defined on special topological spaces. It has applications in the study of convergence of sequences of Sugeno and Choquet integrable functions. Regular Borel measures are important in studying Kolmogorov fractal dimensions (Barnsley, 1988; Mandelbrot, 1983).

As it is well known, artificial neural networks are inspired by the biological nervous system, in particular, the human brain and one of the most interesting characteristics of the human brain is its ability to learn. An application of Lusin's theorem in the study of the approximation properties of neural networks was established (Li *et al.*, 2007) since the learning ability of a neural network is closely related to its approximating capabilities. Although simplified, artificial neural networks can model this learning process by adjusting the weighted connections found between neurons in the neuronal network. This effectively emulates the strengthening and weakening of the synaptic connections found in our brains. All these enable the network to learn.

2. Results and Discussions

Hausdorff and Wijsman topologies are hit-and-miss hypertopologies (Apreutesei, 2003; Beer, 1993; Gavriluț & Apreutesei, 2016; Kunze *et al.*, 2012; Hu & Papageorgiou, 1997 (Ch. 1); Precupanu *et al.*, 2016 (Ch. 1); Apreutesei in Precupanu *et al.*, 2006 (Ch. 8); di Maio & di Naimpally, 2008).

Suppose (X,d) is an arbitrary metric space. The *Wijsman topology* τ_W on $\mathcal{P}_0\left(X\right)$ is the supremum $\tau_W^+ \cup \tau_W^-$ of the *upper Wijsman topology* τ_W^+ and the *lower Wijsman topology* τ_W^-, the family

$$\mathcal{F} = \{M \in \mathcal{P}_0(X); d(x,M) < \varepsilon\}_{\substack{x\in X\\ \varepsilon>0}} \cup \{M \in \mathcal{P}_0(X); d(x,M) > \varepsilon\}_{\substack{x\in X\\ \varepsilon>0}}$$

being a subbase for τ_W on $\mathcal{P}_0(X)$.

Remarks 1. I) If $\{M_i\}_{i\in I} \subset \mathcal{P}_0(X)$, the following statements are equivalent:

i) $M_i \xrightarrow{\tau_W} M \in \mathcal{P}_0(X)$;

ii) for every $x \in X$, $d(x,M_i) \xrightarrow{p} d(x,M)$ (pointwise convergence);

iii) $M_i \xrightarrow{\tau_W^+} M$ and $M_i \xrightarrow{\tau_W^-} M$;

II) i) $M_i \xrightarrow{\tau_W^+} M$ iff for every $x \in X$, $\liminf\limits_i d(x,M_i) \ge d(x,M)$ (*i.e.*, for every $0 < \varepsilon < \varepsilon'$ with $S(x,\varepsilon') \cap M = \varnothing$, there is $i_0 \in I$ so that for every $i \in I$, with $i \ge i_0$, we have $S(x,\varepsilon) \cap M_i = \varnothing$).

ii) $M_i \xrightarrow{\tau_W^-} M$ iff for every $x \in X$, $\limsup\limits_i d(x,M_i) \le d(x,M)$ (*i.e.*, for every $D \in \tau_d$ with $D \cap M \ne \varnothing$, there is $i_0 \in I$ so that for every $i \in I$, with $i \ge i_0$ we have $D \cap M_i \ne \varnothing$).

III) (Gavriluț & Apreutesei, 2016) i) If (X,d) is a complete, separable metric space, then the family $\mathcal{P}_f(X)$ of all non-empty closed subsets of X endowed with the Wijsman topology is a Polish space (Beer, 1993). Moreover, the space $(\mathcal{P}_f(X), \tau_W)$ is Polish iff (X,d) is Polish.

ii) (X,d) is separable iff $\mathcal{P}_f(X)$ is either metrizable, first-countable or second-countable. The dependence of the Wijsman topology on the metric d is quite strong. Even if two metrics are uniformly equivalent, they may generate different Wijsman topologies.

IV) If $M \in \mathcal{P}_0(X), \{x_1, x_2, ..., x_n\} \subset X$ and $\varepsilon > 0$ are arbitrarily chosen, then τ_W^- is generated by the family

$U_W^-(M, x_1, x_2, ..., x_n, \varepsilon) = \{N \in \mathcal{P}_0(X); d(x_i, N) < d(x_i, M) + \varepsilon, \forall$
$i = \overline{1, n}\}$, while τ_W^+ is generated by the family

$U_W^+(M, x_1, x_2, ..., x_n, \varepsilon) = \{N \in \mathcal{P}_0(X); d(x_i, M) < d(x_i, N) + \varepsilon, \forall\, i = \overline{1, n}\}$.

The *Hausdorff-Pompeiu pseudo-metric* h on $\mathcal{P}_f(X)$ is defined by

$$h(M, N) = \max\{e(M, N), e(N, M)\},$$

$\forall\, M, N \in \mathcal{P}_f(X)$, where $e(M, N) = \sup_{x \in M} d(x, N)$ is *the excess* of M over N and $d(x, N) = \inf_{y \in N} d(x, y)$ is the distance from x to N (with respect to the metric d).

The topology τ_H induced by the Hausdorff pseudo-metric h is called the *Hausdorff-Pompeiu hipertopology on* $\mathcal{P}_f(X)$. On $\mathcal{P}_{bf}(X)$ (the family of all nonvoid bounded closed subsets of X), h becomes a veritable metric. If, in addition, X is complete, then the same is $\mathcal{P}_f(X)$ (Hu & Papageorgiou, 1997).

Let $\mathcal{P}_k(X)$ be the family of all nonvoid compact subsets of X. We observe that $e(N, M) = h(M, N),\ \forall\ M, N \in \mathcal{P}_f(X),$ with $M \subseteq N.$ Also, $e(M, N) \le e(M, P),\ \forall\ M, N, P \in \mathcal{P}_f(X),$ with $P \subseteq N$ and $e(M, P) \le e(N, P),\ \forall\ M, N, P \in \mathcal{P}_f(X),$ with $M \subseteq N.$ Generally, even if $M, N \in \mathcal{P}_k(X),$ then $e(M, N) \ne e(N, M).$ If $M \in \mathcal{P}_f(X)$ and $\varepsilon > 0$, let be

$S(M, \varepsilon) = \{x \in X; \exists m \in M, d(x, m) < \varepsilon\} (= \bigcup_{m \in M} \{x \in X; m \in M, d(x, m) < \varepsilon\}),$

Since $h(M, N) < \varepsilon$ iff $M \subset S(N, \varepsilon)$ and $N \subset S(M, \varepsilon)$, we get:

$$h(M, N) = \inf\{\varepsilon > 0; M \subset S(N, \varepsilon), N \subset S(M, \varepsilon)\}.$$

In other words, $\tau_H = \tau_H^+ \cup \tau_H^-$ is the supremum of the *upper Hausdorff topology* τ_H^+ (having as a base family $\{U^+(M, \varepsilon)\}_{\varepsilon > 0}$, where $U^+(M, \varepsilon) = \{N \in \mathcal{P}_f(X); N \subset S(M, \varepsilon)\}$ and the *lower Hausdorff topology*

τ_H^- (having as a base the family $\{U^-(M,\varepsilon)\}_{\varepsilon>0}$, where $U^-(M,\varepsilon)=\{N\in\mathcal{P}_f(X);\quad M\subset S(N,\varepsilon)\}$) .

Another equivalent expression of the Hausdorff distance between two sets $M,N\in\mathcal{P}_f(X)$ is:

$$h(M,N)=\sup\{|\,d(x,N)-d(x,M)\,|;x\in X\}.$$

From here, the uniform aspect of the Hausdorff topology derives - it is the topology on $\mathcal{P}_f(X)$ of uniform convergence on X of the distance functionals $x\mapsto d(x,M)$, with $M\in\mathcal{P}_f(X)$. Hausdorff topology is invariant with respect to uniformly equivalent metrics (Apreutesei, 2003).

Remarks 2. (Apreutesei, 2003; Gavriluț & Apreutesei, 2016; Precupanu *et al*., 2016 (Ch. 1); Apreutesei in Precupanu *et al*., 2006 (Ch. 8)).

i) If one replaces the pointwise convergence of Wijsman convergence by the uniform convergence (in x), then Hausdorff convergence induced by the Hausdorff pseudometric is obtained. Generally, Hausdorff topology τ_H is finer than Wijsman topology τ_W. Hausdorff and Wijsman topologies on $\mathcal{P}_f(X)$ are coincident iff (X,d) is totally bounded.

ii) If X is a real normed space, then, on the class of monotone sequences of subsets of $\mathcal{P}_k(X)$, Hausdorff and Wijsman topologies are equivalent.

iii) Hausdorff metric on $\mathcal{P}_k(X)$ is an essential tool in the study of fractals, hyperfractals, multifractals and superfractals – (Andres & Fišer, 2004; Andres & Rypka, 2012; Kunze *et al*., 2012).

The space $(\mathcal{P}_k(X),h)$ is called "the life space of fractals" (Barnsley, 1988). In a more general setting than using Hausdorff topology, a fractal approach has recently been proposed using Vietoris topology (Banakh & Novosad, 2013).

In what follows, we introduce regularity properties. An appropriate framework is the following:

Let T be a locally compact, Hausdorff space, $\mathcal{C}$ a ring of subsets of T and X a real normed space. For instance, $\mathcal{C}$ might be the Baire δ-ring $\mathcal{B}_0$ (respectively, the Baire σ-ring $\mathcal{B}_0'$) generated by the lattice of all compact subsets of T which are G_δ (*i.e.*, countable intersections of open sets) or $\mathcal{C}$ is the Borel δ-ring $\mathcal{B}$ (respectively, the Borel σ-ring $\mathcal{B}'$) generated by the lattice $\mathcal{K}$ of all compact subsets of T. Obviously, $\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{B}'$ and $\mathcal{B}_0 \subset \mathcal{B}_0'$. If T is metrisable or if it has a countable base, then any compact set $K \subset T$ is G_δ. In this case, $\mathcal{B}_0 = \mathcal{B}$ (Dinculeanu, 1964, Ch. III, p. 187), so $\mathcal{B}_0' = \mathcal{B}'$.

Regularity can be viewed as a continuity property with respect to a suitable topology on $\mathcal{P}(T)$ (Dinculeanu, 1964, Ch. III, p. 197):

In the locally compact Hausdorff space T we denote by $\mathcal{D}$ the family of all open subsets of T. and by $\mathcal{I}(K,D) = \{A \subset T \,/\, K \subset A \subset D\}$, $\forall K \in \mathcal{K}, \forall D \in \mathcal{D}$, with $K \subset D$.

We observe that $\mathcal{I}(K,D) \cap \mathcal{I}(K',D') = \mathcal{I}(K \cup K', D \cap D')$, $\forall \mathcal{I}(K,D)$, $\mathcal{I}(K',D')$. In consequence, the family $\{\mathcal{I}(K,D)\}_{\substack{K\in\mathcal{K}\\ D\in\mathcal{D}}}$ is a base of a topology $\tilde{\tau}$ on $\mathcal{P}(T)$. We denote by $\tilde{\tau}$, the topology induced on any subfamily $\mathcal{S} \subset \mathcal{P}(T)$ of subsets of T. By $\tilde{\tau}_l$ (respectively, $\tilde{\tau}_r$) we denote the topology induced on $\{\mathcal{I}(K)\}_{K\in\mathcal{K}} = \{\{A \subset T \,/\, K \subset A\}\}_{K\in\mathcal{K}}$ (respectively, $\{\mathcal{I}(D)\}_{D\in\mathcal{D}} = \{\{A \subset T \,/\, A \subset D\}\}_{D\in\mathcal{D}}$) (Dinculeanu, 1964, Ch. III, pp. 197−198).

Definition 3. A class $\mathcal{F} \subset \mathcal{P}(T)$ is *dense* in $\mathcal{P}(T)$ with respect to the topology induced by $\tilde{\tau}$ if for every $K \in \mathcal{K}$ and every $D \in \mathcal{D}$, with $K \subset D$, there is $A \in \mathcal{C}$ such that $K \subset A \subset D$.

Remarks 4. i) $\mathcal{B}_0, \mathcal{B}, \mathcal{B}_0', \mathcal{B}'$ are all dense in $\mathcal{P}(T)$ with respect to the topology induced by $\tilde{\tau}$.

ii) For every $A \in \mathcal{C}$, there always exists $D \in \mathcal{D} \cap \mathcal{C}$ so that $A \subset D$.

iii) If $\mathcal{C} = \mathcal{B}$ or $\mathcal{B}'$, then for every $A \in \mathcal{C}$, there exist $K \in \mathcal{K} \cap \mathcal{C}$ and $D \in \mathcal{D} \cap \mathcal{C}$ such that $K \subset A \subset D$.

(these statements can be easily verified since T is locally compact (Dinculeanu, 1964, Ch. III, p. 197)).

Definition 5. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is *monotone* (with respect to the inclusion of sets) if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$ with $A \subseteq B$.

Example 6. Suppose X is an *AL*-space (*i.e.*, a real Banach space equipped with a lattice order relation, which is compatible with the linear structure of X, such that the norm $\|\cdot\|$ on X is monotone, that is, $|x| \le |y|$ implies $\|x\| \le \|y\|$, for every $x, y \in X$, and also satisfying the supplementary condition $\|x+y\| = \|x\| + \|y\|$, for every $x, y \in X$, with $x, y \ge 0$).

($\mathbb{R}, L_1(\mu), l_1$ are some examples of *AL*-spaces).

Let Λ be the positive cone of X. By $[x,y]$ we mean the interval consisting of all $z \in X$ so that $x \leq z \leq y$. Suppose $m : \mathcal{C} \to \Lambda$ is an arbitrary set function, with $m(\varnothing) = 0$. We consider *the induced set multifunction* $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ defined for every $A \in \mathcal{C}$ by $\mu(A) = [0, m(A)]$. We observe that $h(\mu(A), \{0\}) = \sup\limits_{0 \leq x \leq m(A)} \|x\| = \|m(A)\|$, for every $A, B \in \mathcal{C}$. If, particularly, $X = \mathbb{R}$ and $m : \mathcal{C} \to \mathbb{R}_+$ is an arbitrary set function, with $m(\varnothing) = 0$, then the induced set multifunction is $\mu : \mathcal{C} \to \mathcal{P}_{bf}(\mathbb{R})$ defined for every $A \in \mathcal{C}$ by $\mu(A) = [0, m(A)] (\subset \mathbb{R})$. We observe that μ is monotone if and only if the same is m.

In what follows, let $\mu : (\mathcal{C}, \tau_1) \to (\mathcal{P}_f(X), \tau_2)$ be a monotone set multifunction, where $\tau_1 \in \{\tilde{\tau}, \tilde{\tau}_l, \tilde{\tau}_r\}$ and $\tau_2 \in \{\tau_H, \tau_W\}$ $\tau_2 = \tau_2^+ \cup \tau_2^-$, where $\tau_2^+ \in \{\tau_H^+, \tau_W^+\}$ and $\tau_2^- \in \{\tau_H^-, \tau_W^-\}$. Let $\mathcal{B}_1 \in \{\{\mathcal{I}(K,D)\}_{\substack{K \in \mathcal{K} \\ D \in \mathcal{D}}}, \{\mathcal{I}(K)\}_{K \in \mathcal{K}}, \{\mathcal{I}(D)\}_{D \in \mathcal{D}}\}$, be a base for τ_1 and $\mathcal{B}_2$ be a base for τ_2.

Definition 7. A set $A \in \mathcal{C}$ is (τ_2-) *regular* if $\mu : (\mathcal{C}, \tau_1) \to (\mathcal{P}_f(X), \tau_2)$ is continuous at A, *i.e.* for every $(A_i)_{i \in I}, A \subset \mathcal{C}$, with $A_i \xrightarrow{\tau_1} A$, we have $\mu(A_i) \xrightarrow{\tau_2} \mu(A)$.

When τ_1 is $\tilde{\tau}$, or, respectively $\tilde{\tau}_l$ or $\tilde{\tau}_r$, we get the notions of (τ_2-) *regularity*, (τ_2-)R_l- *regularity* (also called *inner regularity*) or (τ_2-)R_r-*regularity* (also called *outer regularity*).

The statements below easily follow:

Proposition 8. An arbitrary set $A \in \mathcal{C}$ is:

i) regular if and only if for every $\mathcal{V} \in \mathcal{B}_2$, with $\mu(A) \in \mathcal{V}$, there exist $K \in \mathcal{K} \cap \mathcal{C}, K \subset A$ and $D \in \mathcal{D} \cap \mathcal{C}, D \supset A$ so that for every $B \in \mathcal{C}$, with $K \subset B \subset D$, we have $\mu(B) \in \mathcal{V}$;

ii) R_l-regular if and only if for every $\mathcal{V} \in \mathcal{B}_2$, with $\mu(A) \in \mathcal{V}$, there exists $K \in \mathcal{K} \cap \mathcal{C}, K \subset A$ so that for every $B \in \mathcal{C}$, with $K \subset B \subset A$, we have $\mu(B) \in \mathcal{V}$;

iii) $R_r -$ regular if and only if for every $\mathcal{V} \in \mathcal{B}_2$, with $\mu(A) \in \mathcal{V}$, there exists $D \in \mathcal{D} \cap \mathcal{C}, D \supset A$ so that for every $B \in \mathcal{C}$, with $A \subset B \subset D$, we have $\mu(B) \in \mathcal{V}$.

Remark 9. I) Any $K \in \mathcal{K}$ is $R_l -$regular; any $D \in \mathcal{D}$ is $R_r -$regular.

II) For $\tau_2 = \tau_H$ or τ_W we obtain the notions of regularity from (Gavriluţ & Apreutesei, 2016; Gavriluţ, 2010; Gavriluţ, 2012). For instance, if $\tau_2 = \tau_H$, then as a consequence of the monotonicity of μ, we have the following expressions of regularity as an approximation property (in the sense of Gavriluţ, 2010):

i) *regular* if for every $\varepsilon > 0$, there are $K \in \mathcal{K} \cap \mathcal{C}, K \subset A$ and $D \in \mathcal{D} \cap \mathcal{C}, D \supset A$ so that $h(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}$, with $K \subset B \subset D$.

ii) R_l – *regular* if for every $\varepsilon > 0$, there exists $K \in \mathcal{K} \cap \mathcal{C}, K \subset A$ so that $h(\mu(A), \mu(B)) = e(\mu(A), \mu(B)) < \varepsilon$, for every $B \in \mathcal{C}$, with $K \subset B \subset A$.

iii) R_r – *regular* if for every $\varepsilon > 0$, there exists $D \in \mathcal{D} \cap \mathcal{C}, D \supset A$ such that $h(\mu(A), \mu(B)) = e(\mu(B), \mu(A)) < \varepsilon$, for every $B \in \mathcal{C}$, with $A \subset B \subset D$.

III) i) μ is regular if and only if for every $\varepsilon > 0$, there are $K \in \mathcal{K} \cap \mathcal{C}, K \subset A$ and $D \in \mathcal{D} \cap \mathcal{C}, D \supset A$ so that $e(\mu(D), \mu(K)) < \varepsilon$;

ii) μ is R_l -regular if and only if for every $\varepsilon > 0$, there is $K \in \mathcal{K} \cap \mathcal{C}, K \subset A$ so that $e(\mu(A), \mu(K)) < \varepsilon$;

iii) μ is R_r -regular if and only if for every $\varepsilon > 0$, there is $D \in \mathcal{D} \cap \mathcal{C}, D \supset A$ so that $e(\mu(D), \mu(A)) < \varepsilon$.

In what follows, regularity and fractality are considered from a physical perspective. In this sense, we present some physical correspondences with hit-and-miss topologies. Several regularizations by sets of functions of ε - approximation type scale and their implications will be also provided.

The same as some physical concepts, hit-and-miss hypertopologies become consistent when considered together, although they are composed of two independent parts - the upper and the lower hypertopologies. For example, in physical terms, the non-differentiability of the curve motion of the physical object involves the simultaneous definition at any point of the curve, of two differentials (left and right). Since we cannot favor one of the two differentials, the only solution is to consider them simultaneously through a complex differential (its application, multiplied by dt, where t is an affine parameter, to the field of space coordinates implies complex velocity fields).

If we consider a fractal function $f(x)$, with $x \in [a,b]$ (one of the trajectory's equation) and the sequence of the values of the variable x:

$$x_a = x_0, x_1 = x_0 + \varepsilon, ..., x_k = x_0 + k\varepsilon, ..., x_n = x_0 + n\varepsilon = x_b. \quad (1)$$

Then by $f(x,\varepsilon)$, we denote the fractured line connecting the points

$$f(x_0), ..., f(x_k), ..., f(x_n).$$

The broken line will be considered as an approximation which is different from the one used before. We shall say that $f(x,\varepsilon)$ is an $\varepsilon -$ *approximation scale.* If we consider the $\bar{\varepsilon} -$ approximation scale $f(x,\bar{\varepsilon})$ of the same function, since $f(x)$ is similar almost everywhere, if ε and $\bar{\varepsilon}$ are small enough, then the two approximations $f(x,\varepsilon)$ and $f(x,\bar{\varepsilon})$ must lead to the same results when we study a fractal phenomenon by approximation. If we compare the two cases, then to an infinitesimal increase $d\varepsilon$ of ε, it corresponds an increase $d\bar{\varepsilon}$ of $\bar{\varepsilon}$, if the scale is dilated.

Simultaneous invariance with respect to both space-time coordinates and the resolution scale induces general Scale Relativity Theory (El-Nabulsi, 2012; El-Nabulsi, 2013). These theories are more general than Einstein's general Relativity Theory, being invariant with respect to the generalized Poincaré group (standard Poincaré group and dilatation group) (El-Nabulsi, 2012; El-Nabulsi, 2013). Basically, we discuss various physical theories built on manifolds of fractal space-time. They turn out to be reducible to one of the following classes:

i) Scale Relativity Theory (Nottale, 1993; Nottale, 2011) and its possible extensions (El Naschie *et al.*, 1995). It is assumed that the motion of microparticles takes place on continuous but non-differentiable curves. In such context, regularization works using sets of functions of $\varepsilon -$ approximation type scale;

ii) Transition in which to each point of the motion trajectory, a transfinite set is assigned (in particular, a Cantor type set (El Naschie *et al.*, 1995) $\varepsilon^{(\infty)}$ model of space-time), in order to mimic the continuous (the trans-physics). In such context, the regularization of "vague" sets by known sets works;

iii) Fractal string theories containing simultaneously relativity and trans-physics (Hawking & Penrose, 1996; Penrose, 2004). The reduction of the complex dimensions to their real part is equivalent to Scale Relativity Type theories, while reducing them to the imaginary part of their complex dimensions generates trans-physics. In such context, the simultaneous regularization by sets of functions of $\varepsilon-$approximation type scale and also by “known” sets works. The “reduction” of the complex dimensions to their real part requires the regularization by sets of functions of $\varepsilon-$approximation type scale, while the “reduction” to their imaginary part requires regularization with “known” sets. We now assume that the complex system particle moves on continuous, but non-differentiable curves (fractal curves). It is well-known that the nervous influx (through brains neuronal network) takes place on continuous but non-differentiable curves in the hydrodynamics variant of scale relativity (with arbitrary constant fractal dimension). In this way, the complex systems dynamics can be simplified and all physical phenomena involved depend not only on space-time coordinates but also on space-time scale resolution. That is why physical quantities describing the complex systems dynamics can be considered as fractal functions. Once accepted such a hypothesis, some consequences of non-differentiability by Scale Relativity Theory (SRT) are evident (Nottale, 1993; Nottale, 2011; Gavriluț & Agop, 2015a). In such perspective, some applications of non-differentiability in biological systems were recently given by Stoica *et al.*, 2015; Duceac *et al.*, 2015a; Duceac *et al.*, 2015b; Doroftei *et al.*, 2016; Nemeş *et al.*, 2015; Postolache *et al.*, 2016; Ştefan *et al.*, 2016.

i) Physical quantities that describe the complex system are fractal functions, *i.e.*, functions depending both on spatial coordinates and time as well as on the scale resolution, $\frac{\delta t}{\tau}$ (identified with $\frac{dt}{\tau}$ by substitution principle (Nottale, 1993, Nottale, 2011)). In classical physics, the physical quantities describing the dynamics of a complex system are continuous, but differentiable functions depending only on spatial coordinates and time;

ii) Two complementary representations result: the formalism of the fractal hydrodynamics (at the continuum level) on one hand, and Schrödinger’s type theory (at the discontinuum level) on the other hand. Moreover, the chaoticity, either through turbulence in the fractal hydrodynamic approach, or through stochasticization in the Schrödinger type approach, is generated only by the non-differentiability of the motion trajectories in a fractal space. We note that some consequences of the turbulence in biological systems have been recently given by Duceac, 2015c; Duceac *et al.*, 2016; Păvăleanu *et al.*, 2016, Velenciuc *et al.*, 2016; Duceac *et al.*, 2017.

3. Conclusions

A mathematical-physical perspective on fractality, regularity and several hit-and-miss hypertopologies is considered. In future works, we aim to develop a neuronal network fractal theory using Wijsman topology, since its pointwise character could characterize some properties better than Hausdorff topology does.

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PROPRIETĂŢI DE APROXIMARE DIN PERSPECTIVĂ FIZICO-MATEMATICĂ. POSIBILE CORELAŢII CU FRACTALITATEA REŢELELOR NEURONALE

(Rezumat)

În această lucrare, sunt abordate diferite probleme referitoare la proprietatea de regularitate, văzută ca o proprietate de aproximare în unele hipertopologii de tip loveşte-şi-lasă. Punem astfel bazele pentru autosimilaritate şi fractalitate din perspectivă fizico-matematică. Întrucât în unele exemple de fractali, cum ar fi reţeaua neuronală sau sistemul circulator, proprietatea uniformă a hipertopologiei Hausdorff nu este potrivită, hipertopologia Wijsman ar putea fi preferată, putând descrie mai bine proprietăţile punctuale pe care fractalii le posedă.