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SOME REMARKS ON THE CLASSIFICATION OF THE WEYL CONFORMAL TENSOR IN 4-DIMENSIONAL MANIFOLDS OF NEUTRAL SIGNATURE

BY

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Abstract. This paper presents a brief discussion of the algebraic classification of the Weyl conformal tensor on a $4-$dimensional manifold with metric g of neutral signature $(+,+,-,-)$. The classification is algebraically similar to the well-known Petrov classification in the Lorentz case and the various algebraic types and corresponding canonical forms are obtained. Further details on principal, totally null $2-$spaces and null directions similar to those of L. Bel in the Lorentz case are described.

Keywords: Weyl tensor classification; neutral signature; algebraic structures.

1. Introduction

Let M be a $4-$dimensional manifold with smooth metric of neutral signature $(+,+,-,-)$ and let C be the Weyl conformal tensor for (M,g). The idea is to provide an algebraic classification of C similar to that given by Petrov in the Lorentz case. The discussion here is brief and more details will be given elsewhere (Hall, 2017). After this work was completed the author was

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informed that ideas similar to some of those reported here have been given in (Law, 1991; Law, 2006; Batista, 2013; Ortaggio, 2009) and another approach was also presented in (Coley and Hervik, 2010). However, the work here is claimed to be simpler, more structured and to go much further and is more amenable for purposes of calculation.

2. Algebraic and Geometric Preliminaries

At $m \in M$ the tangent space to M, $T_m M$, has a basis x, y, s, t satisfying $x \cdot x = y \cdot y = -s \cdot s = -t \cdot t = 1$ (where $\cdot$ denotes an inner product with respect to $g(m)$) and an associated null basis of (null) vectors l, n, L, N at m given by $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that $l \cdot n = L \cdot N = 1$ with all other such inner products zero. The space of all $2-$forms (*bivectors*) at m is denoted by $\Lambda_m M$ and is a Lie algebra under matrix commutation. A bivector F has matrix rank either 2 or 4 and, if it is 2, F is called *simple*. A simple bivector may be written in components as $F^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_m M$ and the $2-$dimensional subspace of $T_m M$ spanned by u, v is uniquely determined by F and called the *blade* of F. Now, with $*$ denoting the usual duality operator and for $E \in \Lambda_m M$ one has ${}^*E^* = E$ and one may define the subalgebras $\overset{+}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = E\}$ and $\overset{-}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = -E\}$ of $\Lambda_m M$. Each member of $\Lambda_m M$ may be uniquely decomposed into the sum of members of $\overset{+}{S}_m$ and $\overset{-}{S}_m$. One also has a metric P on $\Lambda_m M$ given for $E, E' \in \Lambda_m M$ by $P(E, E') = E^{ab} E'_{ab}$ and this metric has signature $(+,+,-,-,-,-)$. It then follows that if $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$, $P(\overset{+}{E}, \overset{-}{E}) = 0$ and P restricts to a metric of Lorentz signature $(+,-,-)$ on each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$. This leads to the Lie algebra product $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$. Each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is Lie-isomorphic to $o(1,2)$ and, of course, $\Lambda_m M$ is Lie-isomorphic to $o(2,2)$. Particularly important simple members of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are the *totally null bivectors* (and they are the only simple members of $\overset{+}{S}_m$ and $\overset{-}{S}_m$) whose blades are spanned by an orthogonal pair of null members of $T_m M$. Choosing an orientation for $T_m M$ one may then choose a null basis for $T_m M$, as above, and then a basis F, G, H for $\overset{+}{S}_m$ where $F = l \wedge n - L \wedge N$, $G = l \wedge N$

and $H = n \wedge L$ (and similarly $\bar{F} = l \wedge n + L \wedge N$, $\bar{G} = l \wedge L$ and $\bar{H} = n \wedge N$ is a basis for $\bar{S}_m$). In these bases G and H are totally null members of $\overset{+}{S}_m$ and $\bar{G}$ and $\bar{H}$ are totally null members of $\bar{S}_m$.

3. The Weyl Tensor Classification

The Weyl conformal tensor C for (M, g) satisfies ${}^{*}C = C^{*}$ and may be decomposed at any $m \in M$ into tensors $\overset{+}{W}$ and $\overset{-}{W}$ as

$$C = \overset{+}{W} + \overset{-}{W} \qquad \overset{+}{W} \equiv \frac{1}{2}(C + {}^{*}C), \qquad \overset{-}{W} \equiv \frac{1}{2}(C - {}^{*}C) \tag{1}$$

Thus $\overset{+}{W}{}^{*} = \overset{+}{W}$ and $\overset{-}{W}{}^{*} = -\overset{-}{W}$. Next consider the linear map f on bivectors at m given by $f : E^{ab} \to C^{ab}{}_{cd} E^{cd}$ together with maps $\overset{+}{f}$ and $\overset{-}{f}$ obtained in a similar way from $\overset{+}{W}$ and $\overset{-}{W}$. The subspaces $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are invariant subspaces of f. Now the map $\overset{+}{f} : \overset{+}{S}_m \to \overset{+}{S}_m$ is a linear map on a $3-$dimensional space of Lorentz signature and may be algebraically classified into its Jordan forms (Segre types) and the only types which arise are $\{111\}$ (diagonable over ℝ), $\{1z\bar{z}\}$ (diagonable over ℂ), $\{21\}$ (eigenvalues real) and $\{3\}$ (with eigenvalue zero from the tracefree condition on $\overset{+}{W}$ which follows from that on C). Using the basis for $\overset{+}{S}_m$ given above it can be shown that the above four Jordan types for $\overset{+}{f}$ (that is, for $\overset{+}{W}$) give the following "canonical" forms for $\overset{+}{W}(m)$)

$$\overset{+}{W}_{abcd}(m) = \frac{\rho_1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} + F_{ab}F_{cd}) + \frac{\rho_2}{2}(G_{ab}G_{cd} \pm H_{ab}H_{cd}) \tag{2}$$

$$\overset{+}{W}_{abcd}(m) = \frac{\rho_1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} + F_{ab}F_{cd}) \pm G_{ab}G_{cd} \tag{3}$$

$$\overset{+}{W}_{abcd}(m) = (G_{ab}F_{cd} + F_{ab}G_{cd}) \tag{4}$$

for $\rho_1, \rho_2 \in \mathbb{R}$. By analogy with the Petrov classification of $C(m)$ in the Lorentz case (Petrov, 1969) (and cf (Hall, 2004)), call $\overset{+}{W}(m)$ in Eq. (2) type $\mathbf{I}$ if the eigenvalues are distinct. If two eigenvalues are equal in Eq. (2) (Segre type $\{1(11)\}$) there are two possibilities; first when the resulting eigen-2-space of bivectors has Lorentz signature in $\overset{+}{S}_m$ ($\rho_2 = 0$ in Eq. (2)) and this type is called $\mathbf{D_1}$ and second when this eigen-2-space is Euclidean ($3\rho_1 = \rho_2 \neq 0$ in Eq. (2)) and this type will be labelled $\mathbf{D_2}$. These are the "degenerate" possibilities for type $\mathbf{I}$. Similarly call $\overset{+}{W}(m)$ in Eq. (3) type $\mathbf{II}$ (and call the degenerate case when the eigenvalue $\rho_1 = 0$ type $\mathbf{N}$). For Eq. (4) the type is labelled $\mathbf{III}$. The degenerate types are thus

$$\begin{aligned}\overset{+}{W}_{abcd}(m) &= \frac{\rho_1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} + F_{ab}F_{cd}) \\ &= \frac{\rho_1}{2}(2\overset{+}{P}_{abcd} + \frac{3}{2}F_{ab}F_{cd}) \qquad (\textit{type } \mathbf{D_1}; \rho_1 \neq 0)\end{aligned} \tag{5}$$

$$\begin{aligned}\overset{+}{W}_{abcd}(m) &= \frac{\rho_1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} + F_{ab}F_{cd}) + \frac{3\rho_1}{2}(G_{ab}G_{cd} + H_{ab}H_{cd}) \\ &= -\rho_1(2\overset{+}{P}_{abcd} - \frac{3}{2}K_{ab}K_{cd}) \qquad (\textit{type } \mathbf{D_2}; \rho_1 \neq 0)\end{aligned} \tag{6}$$

$$\overset{+}{W}_{abcd}(m) = \pm G_{ab}G_{cd} \qquad (\textit{type } \mathbf{N}) \tag{7}$$

where $K \equiv G + H$ and $\overset{+}{P}_{abcd} \equiv \frac{1}{2}(G_{ab}H_{cd} + H_{ab}G_{cd} - \frac{1}{2}F_{ab}F_{cd})$. Finally one adds the type $\mathbf{O}$ at m when $\overset{+}{W}(m) = 0$.

4. Principal Null Directions and Totally Null 2-Spaces

For $\overset{+}{W}(m) \neq 0$ consider the following relationships for a non-zero $k \in T_m M$, a *totally null* bivector $E \in \overset{+}{S}_m$, a non-zero bivector $P \in \overset{+}{S}_m$ *not proportional* to E and satisfying $E_{ab}P^{ab} = 0$, a $1-$form p which is *neither zero nor parallel to* k and real numbers $\alpha, \beta, \gamma, \delta$ with $\delta \neq 0$.

$$(i)\ \overset{+}{W}_{abcd}\, k^b k^d = \alpha k_a k_c, \qquad (ii)\ \overset{+}{W}_{abcd}\, E^{cd} = \beta E_{ab} \tag{8}$$

$$(i)\ \overset{+}{W}_{abcd}\, k^b k^d = k_a p_c + p_a k_c \qquad (ii)\ \overset{+}{W}_{abcd}\, E^{cd} = \gamma E_{ab} + \delta P_{ab} \tag{9}$$

The vector k in Eq. (8(i)) is necessarily null and will be said to span a *repeated principal null direction of* $\overset{+}{W}(m)$ (a *repeated pnd*) (cf (Bel, 2000; Sachs, 1961; Hall, 2004)). The blade of the totally null bivector E in Eq. (8(ii)) will be called a *repeated principal totally null* $2-$*space (a repeated* $2-$*space) of* $\overset{+}{W}(m)$ (and E is an eigenbivector of $\overset{+}{W}(m)$). The vector k in Eq. (9(i)) can be shown to be necessarily null and will be said to span a *general principal null direction of* $\overset{+}{W}(m)$ (a *general pnd*) [and a set of equivalent conditions on k are (i) that $k_{[e}\overset{+}{W}_{a]bc[d}\, k_{f]}k^b k^c = 0$ where square brackets denote the usual skew-symmetrisation of indices, and (ii) that Eq. (8(i)) is false]. Collectively, repeated and general pnds will be referred to simply as *pnds*. The blade of the bivector E in Eq. (9(ii)) will be called a *general principal totally null* $2-$*space* (*a general* $2-$*space*) *of* $\overset{+}{W}(m)$. Collectively, repeated and general such $2-$spaces are called *principal* $2-$ *spaces* of $\overset{+}{W}(m)$. Assuming that $\overset{+}{W}(m) \neq 0$ the following hold;

Lemma 1

(i) There exists $0 \neq k \in T_m M$ such that $\overset{+}{W}_{abcd}\, k^d = 0$ if and only if $\overset{+}{W}(m)$ is type $\mathbf{N}$. The vector k spans a repeated pnd and may be *any* non-zero member of the totally null blade of the bivector G in Eq. (7) (and only these). The bivector G is the unique totally null member of $\overset{+}{S}_m$ (up to a scaling) satisfying Eq. (8(ii)) and, in fact, $\beta = 0$.

(ii) There exists $0 \neq k \in T_m M$ such that $\overset{+}{W}_{abcd}\, k^b k^d = 0$ if and only if $\overset{+}{W}(m)$ is type $\mathbf{N}$ or $\mathbf{III}$. Again k spans a repeated pnd and may be *any* non-zero member of the totally null blade of the bivector G in Eq. (7) or Eq. (4) (and only these). The bivector G is the unique totally null member of $\overset{+}{S}_m$ (up to a scaling) satisfying Eq. (8(ii)) and, in fact, $\beta = 0$.

(*iii*) There exists $0 \neq k \in T_m M$ such that $\overset{+}{W}_{abcd} k^b k^d = \alpha k_a k_c$ with $0 \neq \alpha \in \mathbb{R}$ if and only if $\overset{+}{W}(m)$ is type **II** or $\mathbf{D_1}$. Again k spans a repeated pnd and may be *any* non-zero member of the totally null blade of the bivector G in Eq. (3) for type **II** (and only these), or any member of the totally null blades of G and H in Eq. (5) for $\mathbf{D_1}$ (and only these). The bivectors G (for type **II**) and G and H (for type $\mathbf{D_1}$) are the unique totally null member(s) of $\overset{+}{S}_m$ (up to a scaling) satisfying Eq. (8(*ii*)) and in all cases $\beta \neq 0 \neq \alpha$ with the same β arising for both G and H and the same α for the associated pnds in type $\mathbf{D_1}$.

(*iv*) If there exists $0 \neq k \in T_m M$ such that Eq. (9(*i*)) holds then k spans a general pnd and may be *any* member of the totally null blade of a bivector $E \in \overset{+}{S}_m$ satisfying Eq. (9(*ii*)). The non-zero members of the blade of any totally null $E \in \overset{+}{S}_m$ satisfying Eq. (9(*ii*)) span general pnds.

Thus finding repeated pnds for $\overset{+}{W}(m)$ amounts to finding its totally null eigenbivectors E as in Eq. (8(*ii*)). If such an eigenbivector exists either it is unique (up to a scaling) and then the type of $\overset{+}{W}(m)$ is **N**, **III** ($\beta = 0 = \alpha$ in Eq. (8)) or **II** ($\beta \neq 0 \neq \alpha$ in Eq. (8)) or two independent such eigenbivectors exist each with the same eigenvalue $\beta \neq 0$ ($\Rightarrow \alpha \neq 0$) in Eq. (8) and then the type is $\mathbf{D_1}$. The finding of general pnds amounts to solving Eq. (9(*ii*)) for E and is perhaps more conveniently done by writing this latter equation in the equivalent form $\overset{+}{W}_{abcd} E^{ab} E^{cd} = 0$ with E *not* an eigenbivector of $\overset{+}{W}$. This last equation results in a polynomial equation of order at most 4 for *real* solutions for E. Such solutions can then be calculated from Eq. (2)-Eq. (7). The resulting set of (real) solutions gives the complete set of solutions for principal $2-$spaces and pnds (repeated pnds arising if E is an eigenbivector of $\overset{+}{W}$ and general pnds otherwise) and these solutions can be shown to justify the term “repeated”. It is remarked here that “real” solutions are required. This is because the general solutions of these polynomials sometimes contain complex totally null bivectors as solutions. The blades of such solutions actually contain no non-zero real vectors (up to scaling) and are thus rejected in this analysis (Hall, 2016).

Of course, similar results apply to $\overset{-}{W}$ and $\overset{-}{S}_m$ and the repeated and general pnds collectively give a description of C. To see this consider the following equations for $C(m)$, for a non-zero $k \in T_m M$, for a $1-$form p at m which is *neither zero nor parallel to* k and with $\alpha \in \mathbb{R}$.

$$(i)\ C_{abcd}k^b k^d = \alpha k_a k_c \qquad (ii)\ C_{abcd}k^b k^d = k_a p_c + p_a k_c \tag{10}$$

If $\alpha \neq 0$ in (i), k is necessarily null but this is not true if $\alpha = 0$ (see (Hall, 2017; Hall, 2016)). So suppose that Eq. (10(i)) holds *with* k *assumed null*. Then k is said to span a *repeated principal null direction of* $C(m)$ (a repeated pnd). If Eq. (10(ii)) holds, k is necessarily null (and orthogonal to p) and is said to span a *general principal null direction of* $C(m)$ (a general pnd). [A set of equivalent statements to Eq. (10(ii)) are that (a) $k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0$ at m and (b) that Eq. (10(i)) is false]. Collectively, repeated and general pnds of C are referred to as *pnds* of C. Such directions are related to the analogous ones for $\overset{+}{W}$ and $\overset{-}{W}$ by the following lemma.

Lemma 2

A vector $k \in T_m M$ spans a repeated pnd for C if and only if it spans a repeated pnd for $\overset{+}{W}$ *and* $\overset{-}{W}$. A vector $k \in T_m M$ spans a general pnd for C if and only if it spans a pnd for $\overset{+}{W}$ *and* $\overset{-}{W}$ and is general for at least one of them.

It is noted and easily shown that any real eigenbivector of $C(m)$ is either a member of $\overset{+}{S}_m$ or $\overset{-}{S}_m$ or, if not, lies in an eigenspace of C spanned by eigenbivectors in $\overset{+}{S}_m$ or $\overset{-}{S}_m$. Thus one may think of all the eigenbivectors of C as being in $\overset{+}{S}_m$ or $\overset{-}{S}_m$. In fact, a canonical form for $C(m)$ is obtained from Eq. (1) by simply adding together canonical forms for $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ and the Segre type of $C(m)$ is simply the "sum" of the Segre types of $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ (with any brackets denoting degeneracies appropriately inserted). To determine the pnds of $C(m)$ one notes the following easily checked result that the intersection of two totally null $2-$spaces each of which lies in $\overset{+}{S}_m$ or each of which lies in $\overset{-}{S}_m$ is just the trivial subspace whereas the intersection of two totally null $2-$spaces one of which lies in $\overset{+}{S}_m$ and the other in $\overset{-}{S}_m$ is a null direction at m. Thus when the principal $2-$spaces of $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ are known (and which lie, respectively, in $\overset{+}{S}_m$ and $\overset{-}{S}_m$) their intersections give the pnds of $C(m)$ according to lemma 2. The algebraic type of $C(m)$ can then be labelled $(\mathbf{A},\mathbf{B})$ where $\mathbf{A}$ and $\mathbf{B}$ are the algebraic types for $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$. For example, $C(m)$ has type $(\mathbf{N},\mathbf{N})$ if and only if there exists a unique null direction spanned by k at m satisfying $C_{abcd}k^d = 0$ and which is the intersection of the (unique) repeated principal $2-$spaces for $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ for type $\mathbf{N}$. A consequence of this classification is the fact that there are finitely many (real) principal $2-$spaces for $\overset{+}{W}(m)$ and $\overset{-}{W}(m)$ (possibly none---see an earlier remark) and hence finitely many pnds for $C(m)$ (possibly none) except when the latter's algebraic type is of the form $(\mathbf{A},\mathbf{O})$ for certain choices of $\mathbf{A}$ (*e.g.*, type $(\mathbf{N},\mathbf{O})$) when infinitely many pnds occur.

Of course, the above classification is pointwise on M. However, one can display a topological decomposition of (an open dense subset of) M into *open* subsets of M on which the algebraic types of $\overset{+}{W}$, $\overset{-}{W}$ and C are constant. Also one can demonstrate the local smoothness (in an obvious sense) of the canonical forms and decompositions described in section 3 as well as study the isotropies arising from the the tetrad changes which preserve the given canonical forms for $\overset{+}{W}$, $\overset{-}{W}$ and C. This will be published elsewhere (Hall, 2017). In this last respect the study of the subalgebra structure of $o(2,2)$ given in (Ghanam and Thompson, 2001) and, in a more accessible form for the present purposes in (Wang and Hall, 2013), is useful.

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OBSERVAŢII ASUPRA CLASIFICĂRII TENSORULUI CONFORM WEYL ÎN VARIETĂŢI 4-DIMENSIONALE DE SIGNATURĂ NEUTRĂ

(Rezumat)

Această lucrare prezintă o scurtă discuţie asupra clasificării algebrice a tensorului conform Weyl pe o varietate 4-dimensională cu metrică *g* de signatură neutră (+,+,-,-). Din punct de vedere algebric, clasificarea este similară cu binecunoscuta clasificare Petrov în cazul Lorentz. Sunt obţinute diferite tipuri algebrice şi formele canonice corespunzătoare. Sunt descrise mai multe detalii ale 2-spaţiilor principale total nule şi ale direcţiilor nule, similare celor ale lui L. Bel din cazul Lorentz.