BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Volumul 62 (66), Numărul 3, 2016
Secţia
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

NON-DIFFERENTIABLE PROCESSES

BY

IRINEL CASIAN BOTEZ*

“Gheorghe Asachi” Technical University of Iaşi,
Faculty of Electronics, Telecomunication and Information Technology

Received: November 18, 2016
Accepted for publication: December 19, 2016

Abstract. Our purpose in this paper is to underline the limitations of ordinary differential equations of motion for providing dynamical basis for complex phenomena which justify our approach of using scale relativity theory.

Keywords: Non-differentiable processes; scale relativity theory.

1. Linear Physics

In 1695 Leibniz speculate on the idea of calculating a derivative of order 0.5, in response to a request from L'Hospital. However, in physics, such things as fractional derivatives, have been left aside in favour of long study analytic functions. It was supposed, probably since Galileo, that the physical phenomena can be largely represented by analytic functions and dynamics of physical phenomena can be represented mathematically by equations of motion in which such functions are involved. The truth of this assumption was almost confirmed by Sturm-Liouville Theory success in formalizing acoustic or electromagnetic phenomena, transmission of heat, diffusion or quantum processes. Even when phenomena like phase transitions, turbulence and rheology of polymeric materials could not be explained using the approach

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*Corresponding author; *e-mail*: irinel_casian@yahoo.com

mentioned above, it is believed that the solution can come from a more detailed analysis of the same type.

2. Nonlinear Physics

In the last 45 years, working methodologies in physics have changed, abandoning more and more the linear mathematical methods, analytical and quantitative (Stoyan, 1979; Benoit, 1982; West and Deering, 1994; Schroeder, 2009; Meakin, 2011), heading for a combination of nonlinear mathematical methods, numerical and quantitative (Meakin, 2011). Not only linear methods often have proved inadequate, but even the Euclidean geometry use has not always proved adequate. In any case, Newton formulated the mechanical laws using geometric arguments, although he had developed earlier the differential calculus. Nowhere in "Principia" you will find the famous equation, $\mathbf{F} = m\mathbf{a}$, nor any discussion about derivatives or solving differential equations, but you will find used geometry of Euclid and reports of geometric lengths which are presumed to converge on finite values when the size of timeframes tends to zero. Default felled everywhere in "Principia" is the notion of limit, but nowhere is explicitly discussed this concept. Newton's dynamic arguments are based on the assumption of an absolute space and absolute time, the dimensions are without beginning or end and the continuity is everywhere. The allegations of a continuous and infinite time and space, combined with Euclidean geometry, almost guarantees the continuity of measurements of derived quantities as speed, acceleration and force.

3. Questioning of Differentiability

In the context of modern physics, we learned that the assumptions of absolute space and time are no longer valid and the Euclid's geometry has little to do with physical world. In his lessons about the principles of mechanics, Ludwig Boltzmann (1974), the father of statistical physics, said "… *we have, without apology, presented differentiability as an assumption that agrees with the experimental facts to date*".

But this assumption was invalidated in many cases, of which we evoke here three: a) the problem of turbulence: in 1926, Richardson (Richardson, 1926) published his research on irregular fluctuations in the velocity field of turbulent wind in the atmosphere; b) Levy Statistics: Levy (Levy, 1925; Bologna *et al*., 2002) has set the most general properties needed by a statistical process to violate accepted form, at that time, of the central limit theorem but yet to converge to a limit distribution; c) Brownian motion: the dynamic of this process, developed by Langevin in 1908 using a stochastic differential equation (Levy, 1954) is incompatible with continuous and differentiable nature of the microscopic Hamiltonian dynamics. The differentiability of empirical functions of mechanics is something given by the observational tools which are at our disposal and so nothing precludes, in fact, the use of non-differentiable functions to represent the experimental results. There are at least as many differentiable functions as those non-differentiable. An example of such functions are fractal functions for which an illustrious example being generalized Weierstrass function:

$$F(t)=\sum_{n=-\infty}^{+\infty}\frac{1}{a^n}\sin\left[b^n t+\phi_n\right] \tag{1}$$

which has the derivative:

$$\frac{dF(t)}{dt}=\sum_{n=-\infty}^{+\infty}\left(\frac{b}{a}\right)^n\cos\left[b^n t+\phi_n\right] \tag{2}$$

If b > a, the series (2) diverge in an absolute manner when $n\to\infty$. Note that the function (1) may be either a deterministic fractal function (if the phases $\{\phi_n\}$ are zero), or a stochastic fractal function (in the phases $\{\phi_n\}$ are random).

4. Central Limit Theorem (CLT)

There is a link between the non-differentiability of microscopic processes (Abbott and Wise, 1981), the differentiability of macroscopic processes and the CLT conditions. Let we remind that, according to central limit theorem, if $w_1(t)$, $w_2(t)$, ..., are statistically independent stochastic processes, with identically distribution, than the sum variable:

$$w(t)=\sum_{n=1}^{\infty}w_n(t) \tag{3}$$

is a Gaussian stochastic process.

CLT applies to dynamical systems that have the time scale of microscopic processes much smaller than the time scale for macroscopic processes. In this situation of separation between the two time scales, on long duration, the memory of details given by microscopic dynamics is lost and we can apply a Gaussian statistics on the result at the macroscale. This separation of time scales also means that we can use again ordinary differential calculus at macroscopic scale, even if microscopic dynamics is incompatible with ordinary differential calculus.

5. Two Time Scales

Whether the two time scales are different or not, we must resort to statistical physics. There are two approaches enshrined in this physics. One is the approach of Heisenberg that use dynamic variables. In this approach, in the case of the separation between the two time scales, the transition from microscopic to macroscopic leads to a stochastic differential equation of Langevin type for a macroscopic dynamic variable which corresponds to a random process with Gaussian distribution. The second approach uses Schrodinger's perspective according to which the time evolution of a Liouville density defined in the phase space of the system. The result of this approach is a master equation that usually leads to a conventional diffusion equation, which is a partial differential equation, of second order in relation to space and first-order in relation to time. Therefore, in case of separation of time scales between microscopic and macroscopic, the mathematical description remains in the field of conventional differentiable analytic functions which describe the dynamics (Heisenberg’s approach, centred on particle) or the conventional differential operator (second order partial derivatives in Schrodinger’s approach, centred on the wave). These two approaches were considered equivalent thought for a hundred years, until this equivalence was questionable when different solutions, to the same physical problem, have been obtained using these two approaches (Bologna *et al*., 2002).

When the separation between microscopic time scale and macroscopic time scale is not valid, the memory of non-differentiable nature of the phenomena at the microscopic level is preserved. Thus, transport equations cannot be expressed in terms of ordinary differential calculus, even if our observation is macroscopically.

6. Conclusions

Inability of using macroscopically the ordinary differential calculus is the explanation why the temporal derivative in Langevin equation is replaced by a fractional derivative in relation with time, obtaining a fractional stochastic equation. Likewise, Laplace operator of normal diffusion equation is replaced by a fractional Laplace operator, yielding a fractional diffusion equation in the phase space of the system. We have developed these arguments in (Agop and Casian-Botez, 2015; Agop *et al*. 2015; Casian-Botez and Agop, 2015a; Casian-Botez and Agop, 2015b; Casian-Botez *et al*., 2015).

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PROCESE NEDIFERENȚIABILE

(Rezumat)

Scopul nostru în această lucrare este de a sublinia limitările ecuațiilor diferențiale ordinare de mișcare pentru a pune bazele descrierii dinamici fenomenelor complexe prin utilizarea teoriei relativității de scară.