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NON-STANDARD ANALYSIS AND NON-DIFFERENTIABILITY

BY

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Abstract. In this article we demonstrate the link between fractal features and nediferentiabilitate.

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1. Introduction

In 1821 Cauchy defines the infinitesimals as follows, I quote in French: “On dit qu’une quantité variable deviant *infiniment petite*, lorsque sa valeur numérique décroit indéfiniment de manière à converger vers la limite zéro” (Cauchy, 1821, p. 29). It is good to note that there is a difference between the concepts of *constant decline* and *fall unlimited*. Thus, a variable admitting that successive terms from the following string (Cauchy, 1821):

$$\frac{2}{1},\frac{3}{2},\frac{4}{3},\frac{5}{4},\frac{6}{5},\ldots \qquad (1)$$

extended indefinitely, steadily decreases, but not unlimited, because successive values converge to limit 1. On the other hand, a variable admitting that successive terms from the following string (Cauchy, 1821):

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$$\frac{1}{4},\frac{1}{3},\frac{1}{6},\frac{1}{5},\frac{1}{8},\frac{1}{7},\ldots \tag{2}$$

extended indefinitely, steadily decreases, because the difference between two consecutive terms of this sequence is, alternative, positive and negative; however, the variable decreases unlimited because its values could be made smaller than any given number.

2. Are the Infinitesimals Real Numbers?

The Cauchy's definition of infinitesimals is an intuitive one. Hilbert, in his formal approach to mathematics, is not interested in the question "*what are certain structures?*" but "*what are these properties and what can be deduced from these properties*". Thus, a more accurate definition of the infinitesimals would be: an infinitesimal ε is an element of an ordered field K, nonzero, which has the property:

$$-r < \varepsilon < r \tag{3}$$

whatever positive real number r. It is essential now to mention that any nonzero real number does not check the Eq. (3), so an infinitesimal is not a real number. So, the field should be viewed as an extension of the field $\mathbb{R}$ of the orderly real numbers. This field we call of *hyper-real numbers*, and we will note it by $^*\mathbb{R}$. It is demonstrated (Robinson, 1996) that a finite number k can be put in the form $c+\varepsilon$, where c is a real number, and ε is either zero or an infinitesimal. The real number c is called the standard part of the finite number k, which is written as:

$$\mathrm{c} = \mathrm{st}\left[\mathrm{k}\right] \tag{4}$$

We have been encountered such situations in mathematics. The irrational numbers were introduced in order to solve certain equations. The complex numbers were created by insertion of the ideal item $\sqrt{-1}$. The novelty of the situation with which we are dealing was noted in by Felix Klein. He remarks in Volume 1 of his book (Klein, 1932) that we are dealing with two theories of continuum:

- continuum A (A from Archimedes), illustrated mathematically by the set of real numbers.
- continuum B (B from Bernoulli), mathematical exemplified by what Robinson called *hyperreal numbers*. A possible explanation for the relationship between the two continuums follows. All values from continuum A are (theoretically) possible to be results of measurements. Continuum B has values like $x+dx$ that can never be the reading of measurements.

3. Does it can be Defined the Derivative without Using the Limit?

Leibniz's definition of differential quotient, $\Delta y/\Delta x$, whose logic weakness was criticized by Berkeley, it was amended by Robinson using standard application part, denoted by "st", defined on the continuum B with values in continuum A. With $f(x)$ a function which has values in $^*\mathbb{R}$ and $a < x_0 < b$, where $x_0 \in \mathbb{R}$, we say that it is *differentiable of order 1* if and only if there is a standard real number c for which:

$$f(x) - f(x_0) \simeq c(x - x_0) = c\varepsilon \qquad (5)$$

For any $x \neq x_0$ from the monad of x_0 (Robinson, 1996). The standard real number *c* is named, in this case, the derivative of order 1 of $f(x)$, in x_0 and is denoted by $f^{(1)}(x_0)$ or $f'(x_0)$. So, the derivative can be defined using the *standard part* function:

$$f^{(1)}(x_0) = st\left[\frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon}\right] \qquad (6)$$

where all infinitesimals ε are of order 1 (Robinson, 1996).

4. Conclusions

If expression (6) is constant and independent of ε, f is a differentiable function. But if it is not, Eq. (6) is dependent of infinitesimals ε, which demonstrate that, geometrically, the recommended choice to deal with is a fractal function, $F(x,\varepsilon)$, which depends on ε, but which converge to $f(x)$ when $\varepsilon \to 0$.

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ANALIZA NON-STANDARD ȘI NE-DIFERENȚIABILITATE

(Rezumat)

În acest articol demonstrăm legătura dintre funcțiile fractale și nediferențiabilitate.