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INTEGRABILITY IN INTERVAL-VALUED (SET) MULTIFUNCTIONS SETTING

BY

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Abstract. We recently proposed a new integral of an interval-valued multifunction relative to an interval-valued set multifunction. In this paper we continue the study of this type of integral, establishing various specific properties. Some properties regarding Gould type integrability on atoms are also discussed.

Keywords: gould integral; interval valued (set) multifunction; submeasure; multisubmeasure; non-additive set function; monotone measure.

1. Introduction

The theory of fuzzy sets was introduced by Zadeh (1968). Since then many new theories came out, such as the interval-valued fuzzy sets theory, the generalized theory of uncertainty etc. During the last decade, it has been suggested to use intervals in order to represent uncertainty in the area of decision theory and information theory, for example, calculation of economic uncertainty, theory of interval probability as a unifying concept for uncertainty.

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Thus, a special attention was recently paid to the study of interval-valued (set) multifunctions, since they are related to the representation of uncertainty, a necessity coming from economic uncertainty, fuzzy random variables, interval-probability, martingales of multivalued functions, interval-valued capacities, interval-valued intuitionistic fuzzy sets (Bykzkan and Duan, 2010; Jang, 2004; Jang, 2006; Jang, 2007; Jang, 2011; Jang, 2012; Tan and Chen, 2013; Qin *et al*., 2016; Bustince *et al*., 2013 - in multicriteria decision making problems, Li and Sheng, 1998; Li *et al*., 2014; Weichselberger, 2000 and many others).

Entropy, the well-known concept in physics, information theory and fuzzy set theory, describes the degree of uncertainties and fuzziness of the fuzzy sets. Precisely, entropy and similarity of intuitionistic fuzzy sets are very important in theory and applications in which the intuitionistic fuzzy sets are used to describe the imprecisions and uncertainties.

On the other hand, set-valued functions theory has become an important tool in several practical areas, especially in economic analysis, where it treats problems of individual demand, mean demand, competitive equilibrium, coalition production economies etc. For instance, applications of integration of set-valued functions in economy analysis have roots in Aumann's (1965) research based on the classical Lebesgue integral.

Different types of integrals have been introduced and studied in order to generalize the Riemann integral. In this framework, a way of defining the integral is to use finite or infinite Riemann type sums as in, for instance (Aviles *et al*., 2010; Birkhoff, 1935; Boccuto and Sambucini, 2004; Boccuto and Sambucini, 2012; Boccuto *et al*., 2014; Cascales and Rodriguez, 2004; Dinghas, 1956; Fremlin, 1995; Gould, 1965; Sipos, 1979; Spaltenstein, 1995). An influential work in this direction was Gould's study, 1965, where the concept of an integral using finite sums for real functions with respect to finitely additive vector measures, is introduced.

The Gould integral was generalized and studied in (Gavriluţ and Petcu, 2007a; Gavriluţ and Petcu, 2007b; Gavriluţ *et al.*, 2014) (relative to submeasures), Precupanu and Croitoru, 2002; Precupanu and Croitoru, 2003; Precupanu and Satco, 2008) (relative to multimeasures), (Gavriluţ, 2008; Gavriluţ, 2010; Sofian-Boca, 2011) (relative to multisubmeasures), (Precupanu *et al*., 2010) (relative to monotone set-valued set functions).

A special type of an interval-valued set multifunction was introduced by Sofian-Boca (2011), with respect to the order relation of Guo and Zhang (2004), in order to study a Gould type integral of a real function with respect to it. This type of set multifunction was also investigated by Gavriluţ (2014). In (Jang, 2004; Jang, 2006; Jang, 2007; Jang, 2011; Jang, 2012; Tan and Chen, 2013; Qin *et al*., 2016; Bustince *et al*., 2013) are studied the interval-valued (intuitionistic fuzzy) Choquet integrals and also pointed out their applications in multicriteria decision making problems. Also, in (Hamid and Elmuiz, 2016), the Henstock-Stieltjes integrals of interval-valued functions and fuzzy-number-valued functions are introduced.

In this paper, we continue the study of our new integral (Pap *et al*., submitted for publication) of an interval-valued multifunction relative to an interval-valued set multifunction, pointing out various important properties of it. Indeed, due to this integration type specific, one would expect to obtain special properties.

The paper is organized as follows: Section 1 is for introduction. After Section 2 (of basic concepts, various examples and results), in Section 3 we obtain some remarkable properties of the Gould integral of interval-valued multifunctions relative to a interval-valued set multifunctions. Also, we prove some results regarding Gould type integrability on atoms.

2. Preliminaries

If $n \in \mathrm{N}^*$, by $i = \overline{1,n}$ we mean $i \in \{1,...,n\}$. All over this paper, let be $\mathrm{R}_+ = [0,\infty)$, T a nonempty abstract set, $\mathcal{P}(T)$ the family of all subsets of T and $\mathcal{A}$ an arbitrary algebra of subsets of T.

Definition 1. *A finite partition of* T *is a finite family of nonempty sets* $P = \{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$ *such that* $A_i \cap A_j = \emptyset, i \neq j$ *and* $\bigcup_{i=1}^{n} A_i = T$.

We denote by $\mathrm{P}(T)$ the class of all partitions of T and by P_A, the class of all partitions of A, if $A \in \mathcal{A}$ is fixed.

Definition 2. (*i*) *If* P, $P' \in \mathrm{P}$, P' *is said to be finer than* P *(denoted by* $P \leq P'$ *or* $P' \geq P$*) if every set of* P' *is included in some set of* P.

(ii) *The common refinement of two finite partitions* $P = \{A_i\}_{i=\overline{1,n}}$, $P' = \{B_j\}_{j=\overline{1,m}} \in \mathrm{P}$ *is the partition* $P \wedge P' = \{A_i \cap B_j\}_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$.

Let be an arbitrary set function $m : \mathcal{A} \to \mathrm{R}_+$, with $m(\emptyset) = 0$.

Definition 3. (Pap, 1995) I. m *is said to be:*

(i) *monotone (*or, *fuzzy)* if $m(A) \leq m(B)$, for every $A, B \in \mathcal{A}$, with $A \subseteq B$;

(ii) *subadditive* if $m(A \cup B) \leq m(A) + m(B)$, for every (disjoint) $A, B \in \mathcal{A}$;

(iii) a *submeasure* (in the sense of Drewnowski (1972)) if it is monotone and subadditive;

(iv) *null-additive* if $m(A \cup B) = m(A)$, for every $A, B \in \mathcal{A}$, with $m(B) = 0$;

(v) σ *-subadditive* if $m(A) \leq \sum_{n=0}^{\infty} m(A_n)$, for every (pairwise disjoint) $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, with $A = \bigcup_{n=0}^{\infty} A_n \in \mathcal{A}$;

(vi) *finitely additive* if $m(A \cup B) = m(A) + m(B)$, for every disjoint $A, B \in \mathcal{A}$;

(vii) σ *-additive* if $m(\bigcup_{n=0}^{\infty} A_n) = \sum_{n=0}^{\infty} m(A_n)$, for every pairwise disjoint $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$;

II. A set $A \in \mathcal{A}$ is *an atom with respect to m* if $m(A) > 0$ and for every $B \in \mathcal{A}$, with $B \subset A$, we have either $m(B) = 0$ or $m(A \setminus B) = 0$.

III. m is said to be *finitely purely atomic* if $T = \bigcup_{i=1}^{p} A_i$, where $A_i \in \mathcal{A}$, $i = \overline{1, p}$ are pairwise disjoint atoms of m.

Definition 4. (i) *The variation* $\overline{m}$ *of* m *is the set function* $\overline{m} : \mathcal{P}(T) \to 0, +\infty]$ *defined by* $\overline{m}(E) = \sup\{\sum_{i=1}^{n} m(A_i)\}$, *for every* $E \in \mathcal{P}(T)$, *where the supremum is extended over all finite families of pairwise disjoint sets* $\{A_i\}_{i=1}^{n} \subset \mathcal{A}$, *with* $A_i \subseteq E$, *for every* $i = \overline{1, n}$.

(ii) m is said to be *of finite variation on* $\mathcal{A}$ if $\overline{m}(T) < \infty$.

Remark 5. I. If $E \in \mathcal{A}$, then in the definition of $\overline{m}$ one may consider the supremum over all finite partitions $\{A_i\}_{i=1}^{n} \in \mathcal{P}_E$.

II. $\overline{m}$ is monotone on $\mathcal{P}(T)$.

III. If m is finitely additive, then $\overline{m}(A) = m(A)$, for every $A \in \mathcal{A}$.

IV. If m is subadditive ($\sigma-$ subadditive, respectively) of finite variation, then $\overline{m}$ is finitely additive (σ -additive, respectively) on $\mathcal{A}$.

We denote by $\mathcal{P}_0(\mathbb{R}_+)$ the family of all nonempty subsets of $\mathbb{R}_+$ and by $\mathcal{P}_{kc}(\mathbb{R}_+)$ the family of nonempty, compact convex subsets of $\mathbb{R}_+$.

Definition 6. (Guo and Zhang, 2004) *(the "standard" partial order relation* $'' \preccurlyeq ''$ *on* $\mathcal{P}_0(\mathbb{R}_+)$, *which extends the usual order on* $\mathcal{P}_{kc}(\mathbb{R}_+)$): *If* $A, B \in \mathcal{P}_0(\mathbb{R}_+)$, *then* $A \preccurlyeq B$ *if the following two conditions hold:*

(i) for every $x \in A$, there exists $y_x \in B$ so that $x \leq y_x$;

(ii) for every $y \in B$, there exists $x_y \in A$ so that $x_y \leq y$.

In general, there is no implication between the order relation $'' \preccurlyeq ''$ and the inclusion one. However, on the family $\{[0,a]; 0 \leq a < \infty\}$ they coincide. By convention, $\{0\} = [0,0]$.

If $[a,b],[c,d] \in \mathrm{P}_{kc}(\mathrm{R}_+)$, the following operations are considered (see (Jang, 2007) for details):

I. $[a,b]+[c,d]=[a+b,c+d]$;

II. $\alpha \cdot [a,b]=[\alpha a, \alpha b], \alpha \geq 0$;

III. $[a,b] \cdot [c,d] = [a \cdot c, b \cdot d]$;

IV. $[a,b] \wedge [c,d] = [\min\{a,c\}, \min\{b,d\}]$;

V. $[a,b] \vee [c,d] = [\max\{a,c\}, \max\{b,d\}]$;

VI. $[a,b] \subseteq [c,d]$ if and only if $c \leq a \leq b \leq d$;

VII. $[a,b] \preccurlyeq [c,d]$ if and only if $a \leq c$ and $b \leq d$.

On $\mathrm{P}_0(\mathrm{R})$ we consider the Hausdorff-Pompeiu pseudo-metric h (Hu and Papageorgiou, 1997) defined for every $A, B \in \mathrm{P}_0(\mathrm{R})$ by

$$h(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} |x-y|, \sup_{y\in B}\inf_{x\in A} |x-y|\}.$$

On $\mathrm{P}_{kc}(\mathrm{R}), h$ has the particular form:

$$h([a,b],[c,d]) = \max\{|a-c|,|b-d|\}, \forall a,b,c,d \in \mathrm{R}, a \leq b, c \leq d.$$

According to (Hu and Papageorgiou, 1997), $(\mathrm{P}_{kc}(\mathrm{R}), h)$ is a complete metric space.

For every $M \in \mathrm{P}_{kc}(\mathrm{R}), M = [a,b],$ we denote

$|M| = h(M,\{0\}) (= \max\{|a|,|b|\})$.

If, particularly, $0 < a < c$, then $h([0,a],[0,c]) = c - a$.

If $M \in \mathrm{P}_{kc}(\mathrm{R}_+), M = [a,b],$ then $|M| = b$.

We now recall from (Pap *et al.*, submitted for publication) several notions (in the set-valued case) defined with respect to $'' \preccurlyeq ''$ on $\mathrm{P}_0(\mathrm{R}_+)$:

Definition 7. *Let* $\mu : \mathrm{A} \to \mathrm{P}_0(\mathrm{R}_+)$ *be a set multifunction, with* $\mu(\emptyset) = \{0\}$.

I. μ is said to be:

(i) *an additive multimeasure* if $\mu(A \cup B) = \mu(A) + \mu(B)$, for every disjoint $A, B \in \mathrm{A}$;

(ii) *q-monotone* if $\mu(A) \preccurlyeq \mu(B)$, for every $A, B \in \mathrm{A}$, with $A \subseteq B$;

(iii) *q-subadditive* if $\mu(A \cup B) \preccurlyeq \mu(A) + \mu(B)$, for every disjoint $A, B \in \mathcal{A}$;

(iv) *aq-multisubmeasure* if it is q-monotone and q-subadditive;

(v) *null-additive* if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{A}$, with $\mu(B) = \{0\}$;

II. $A \in \mathcal{A}$ is *a q-atom of* μ if $\{0\} \preccurlyeq \mu(A), \{0\} \neq \mu(A)$ and for every $B \in \mathcal{A}$, with $B \subseteq A$, we have either $\mu(B) = \{0\}$ or $\mu(A \setminus B) = \{0\}$.

III. μ is said to be *finitely purely q-atomic* if $T = \bigcup_{i=1}^{p} A_i$, where $A_i \in \mathcal{A}$, $i = \overline{1, p}$ are pairwise disjoint q-atoms of μ.

Remark 8. Acccording to (Gavriluț, 2014), $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ if and only if there exist two set functions $m_1, m_2 : \mathcal{A} \to \mathbb{R}_+$ so that for every $A \in \mathcal{A}$, $m_1(A) \leq m_2(A)$ (here " $\leq$ " is the usual order on $\mathbb{R}$) and $\mu(A) = [m_1(A), m_2(A)]$. Moreover, in this case, μ is q-monotone, q-subadditive, q-multisubmeasure, null-additive (in the sense of the previous definition) if and only if both m_1, m_2 are monotone, subadditive, submeasures, null-additive, respectively, in the sense of Definition 3.

One can easily generate a q-multisubmeasure (in the sense of (Sofian-Boca, 2011) - see also (Gavriluț, 2014):

Example 9. If $\nu : \mathcal{A} \to \mathbb{R}_+$ is finitely additive, then $m_1, m_2 : \mathcal{A} \to \mathbb{R}_+$, defined by $m_1(A) = \sqrt{\nu(A)}$ and $m_2(A) = \dfrac{\nu(A)}{1 + \nu(A)}$, for every $A \in \mathcal{A}$ are submeasures (Gavriluț, 2012). In consequence, $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$, defined by $\mu(A) = [m_1(A), m_2(A)]$, $\forall A \in \mathcal{A}$ where m_1 and m_2 are as before, is a q-multisubmeasure.

Remark 10. I. A set $A \in \mathcal{A}$ is a q-atom of μ if and only if it is an atom of both m_1 and m_2 (in the sense of Definition 3-II).

II. Suppose μ is null-additive and has q-atoms. If A is a q-atom of μ, then every $B \in \mathcal{A}$, with $B \subseteq A$ and $\{0\} \prec \mu(A)$, is also a q-atom of μ and $\mu(A \setminus B) = \{0\}$, so $\mu(A) = \mu(B)$ also holds.

III. $\overline{\mu} = \overline{m_2}$ (on $\mathcal{A}$).

IV. If μ is a q-multisubmeasure and if $A \in \mathcal{A}$ is a q-atom of μ, then $\overline{\mu}(A) = |\mu(A)| (= m_2(A) = \overline{m_2}(A))$.

3. Gould Integrability

In this section, we point out some remarkable properties of the Gould integral introduced in (Pap *et al*., submitted for publication) for interval-valued multifunctions with respect to an interval-valued set multifunction. Also, we provide different properties regarding Gould type integrability on atoms.

First we recall from (Gould, 1965), the definitions of totally-measurability and Gould integrabiliy of a real function with respect to a set function.

Let $m:\mathcal{A}\to\mathbb{R}_+$ be a non-negative set function, with $m(\emptyset)=0$. Let also $f:T\to\mathbb{R}$ be a real function.

Definition 11. (Gould, 1965) I. f *is said to be* $\overline{m}$*-totally measurable* (*on* $(T,\mathcal{A},m)$) *if for every* $\varepsilon>0$, *there exists* $P_\varepsilon=\{A_i\}_{i=\overline{0,n}}\subset\mathcal{A}, P_\varepsilon\in\mathcal{P}(T)$ *such that:*

(i) $\overline{m}(A_0)<\varepsilon$ and

(ii) $osc(f,A_i)=\sup_{t,s\in A_i}|f(t)-f(s)|<\varepsilon$, for every $i=\overline{1,n}$.

II. f is said to be $\overline{m}$*-totally measurable on* $B\in\mathcal{A}$ if the restriction $f|_B$ of f to B is $\overline{m}$-totally measurable on $(B,\mathcal{A}_B,m_B)$, where $m_B=m|_{\mathcal{A}_B}$ and $\mathcal{A}_B=\{A\cap B; A\in\mathcal{A}\}$.

We consider $\sigma_{f,m}(P)$ (or $\sigma(P)$, for short) $=\sum_{i=1}^{n}f(t_i)m(A_i)$, for every $P=\{A_i\}_{i=\overline{1,n}}\in\mathcal{P}(T)$ and every $t_i\in A_i, i=\overline{1,n}.$

Definition 12. (Gould, 1965) I. f *is said to be Gould* m*-integrable on* T *if the net* $(\sigma(P))_{P\in(\mathcal{P}(T),\leq)}$ *is convergent in* $\mathbb{R}$. *In this case, its limit is called the Gould integral of* f *on* T *with respect to* m*, denoted by* $\int_T fdm.$

II. f *is said to be Gould* m*-integrable on* $B\in\mathcal{A}$ *if the restriction* $f|_B$ *is Gould* m*-integrable on* $(B,\mathcal{A}_B,\mu_B)$.

Remark 13. I. If it exists, the integral of f is unique.

II. f is Gould m-integrable on T if and only if there exists $\alpha\in\mathbb{R}$ such that for every $\varepsilon>0$, there exists $P_\varepsilon\in\mathcal{P}(T)$, so that for every other $P=\{A_i\}_{i=\overline{1,n}}\in\mathcal{P}(T)$, with $P\geq P_\varepsilon$ and every $t_i\in A_i$, $i=\overline{1,n}$, we have $|\sigma(P)-\alpha|<\varepsilon.$

We now recall from (Pap *et al*., submitted for publication) the notions of total-measurability and Gould integrability for an interval-valued multifunction with respect to an interval-valued set multifunction.

All over this section, we suppose that $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is an interval-valued set multifunction, defined by $\mu(A) = [m_1(A), m_2(A)]$, where $m_1, m_2 : \mathcal{A} \to \mathbb{R}_+$, with $m_1(\emptyset) = m_2(\emptyset) = 0$, $m_1(A) \le m_2(A)$, for every $A \in \mathcal{A}$.

Also, let $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ be an interval-valued multifunction, defined by $F(t) = [f_1(t), f_2(t)]$, for every $t \in T$, with $f_1, f_2 : T \to \mathbb{R}_+$, $f_1(t) \le f_2(t)$, for every $t \in T$.

Definition 14. *F is said to be:*

I. *$\overline{\mu}$-totally measurable (on $(T, \mathcal{A}, \mu)$)* if for every $\varepsilon > 0$, there exists $P_\varepsilon = \{A_i\}_{i=\overline{0,n}} \subset \mathcal{A}, P_\varepsilon \in \mathcal{P}(T)$ such that:

(i) $\overline{\mu}(A_0) < \varepsilon$ and

(ii) $osc(F, A_i) = \sup_{t,s \in A_i} h(F(t), F(s)) < \varepsilon$, for every $i = \overline{1,n}$.

II. *$\overline{\mu}$-totally measurable on* $B \in \mathcal{A}$ if $F|_B$ is $\overline{\mu}$-totally measurable on $(B, \mathcal{A}_B, \mu_B)$, where $\mu_B = \mu|_{\mathcal{A}_B}$.

Remark 15. I. If F is $\overline{\mu}$-totally measurable on T, then it is $\overline{\mu}$-totally measurable on every $A \in \mathcal{A}$.

II. F is $\overline{\mu}$-totally measurable on T if and only if the functions f_1, f_2 are $\overline{m_2}$-totally measurable in the sense of (Gould, 1965).

Definition 16. *We denote $\sigma_{F,\mu}(P)$ (or, for short, $\sigma(P)) = \sum_{i=1}^{n} F(t_i)\mu(A_i)$, for every $P = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}(T)$ and every $t_i \in A_i$, $i = \overline{1,n}$.*

I. *F is said to be Gould μ-integrable on T if the net $(\sigma(P))_{P \in (\mathcal{P}, \le)}$ is convergent in $(\mathcal{P}_{kc}(\mathbb{R}), h)$. In this case, its limit is called the Gould integral of F on T with respect to μ, denoted by $\int_T F d\mu$.*

II. *The Gould integral on a subset E of T is defined in classical manner.*

Remark 17. I. If it exists, the integral is unique.

II. If F is μ -integrable on T , then $\int_T F d\mu \in \mathrm{P}_{kc}(\mathrm{R})$, so $\int_T F d\mu = [a,b]$, where $0 \le a \le b$. In consequence, F is μ -integrable on T if and only if there exists $A = [a,b] \in \mathrm{P}_{kc}(\mathrm{R}_+)$ such that for every $\varepsilon > 0$, there exists $P_\varepsilon \in \mathrm{P}$ (T), so that for every $P = \{A_i\}_{i\in\overline{1,n}} \in \mathrm{P}(T)$ with $P \ge P_\varepsilon$ and for every $t_i \in A_i$, $i = \overline{1,n}$, we have $h(\sum_{i=1}^{n} F(t_i)\mu(A_i), A) < \varepsilon$.

III. If $\mu = \{0\}$, then F is μ -integrable on T and $\int_T F d\mu = \{0\}$.

In what follows in this paper, suppose moreover, that $F: T \to \mathrm{P}_{kc}(\mathrm{R}_+)$ is bounded (*i.e.*, there exists $M > 0$ so that $|F(t)| (= f_2(t)) \le M$, for every $t \in T$) and $\mu : \mathrm{A} \to \mathrm{P}_{kc}(\mathrm{R}_+)$ is of finite variation(so $\overline{\mu}(T)(= \overline{m_2}(T)) < \infty)$. We now recall from (Pap *et al*.), some results that will be useful further.

Proposition 18. *F is μ -integrable on T if and only if f_1 is m_1 -integrable and f_2 is m_2 -integrable on T in the sense of (Gavriluț and Petcu, 2007),in this case,*

$$\int_T F d\mu = [\int_T f_1 dm_1, \int_T f_2 dm_2].$$

Theorem 19. *Let $\mu : \mathrm{A} \to \mathrm{P}_{kc}(\mathrm{R}_+)$ be a q-multisubmeasure. The following statements are equivalent:*

I. *F is μ -integrable on T ;*

II. *F is $\overline{\mu}$ -integrable on T ;*

III. *F is $\overline{\mu}$ -totally measurable on T .*

If $F: \mathrm{A} \to \mathrm{P}_{kc}(\mathrm{R}_+)$ is μ -integrable we consider the set multifunction $\varphi : \mathrm{A} \to \mathrm{P}_{kc}(\mathrm{R}_+)$, defined by

$$\varphi(A) = \int_A F d\mu, \forall A \in \mathrm{A}.$$

In (Pap *et al*., submitted for publication) we established some properties of the integral φ . Our aim is to continue this study.

In what follows, we shall understand that *a property holds almost everywhere* (μ *-ae*, for short) if the property holds everywhere excepting a set of null “measure”.

Theorem 20. *Suppose* $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ *is q-monotone. Let* $F,G : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ *be two interval-valued multifunctions such that* F *is* μ *-integrable (on* T *) and* $F = G$ μ *-ae. Then* G *is* μ *-integrable (on* T *) and* $\int_T F d\mu = \int_T G d\mu.$

Proof. Since μ is q-monotone, then by Remark 8, m_1, m_2 are monotone functions. We observe that $F = G$ μ-ae if and only if $f_1 = g_1$ and $f_2 = g_2$ m_2 -ae. In this case, $f_1 = g_1$ and $f_2 = g_2$ m_1 -ae. Since F is μ -integrable, then by Proposition 18, f_1 is m_1 - integrable and f_2 is m_2 -integrable.

Then g_1 is m_1 - integrable, g_2 is m_2 - integrable and $\int_T f_1 dm_1 = \int_T g_1 dm_1$, $\int_T f_2 dm_2 = \int_T g_2 dm_2$ (these statements easily follows by means of the induced set multifunction $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathbb{R}_+)$, $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{A}$, where $m : \mathcal{A} \to \mathbb{R}_+$ is a monotone set function in Theorem 5.3 (Precupanu *et al.*, 2010). In consequence, by Proposition 18, G is μ -integrable. Moreover, $\int_T F d\mu = \int_T G d\mu.$

Theorem 21. *Let* μ *be a q-multisubmeasure and* $F, G : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ *be* μ *-integrable on* T *. Then:*

I. G is φ *-integrableon* T (*where* $\varphi(A) = \int_A F d\mu, \forall A \in \mathcal{A}$);

II. $\int_T G d\varphi = \int_T FG d\mu.$

Proof. Since μ is a q-multisubmeasure, then m_1 and m_2 are submeasures. Let be $\varphi_1 = \int_T f_1 dm_1$ and $\varphi_2 = \int_T f_2 dm_2$. Since by Theorem 19, F is μ-integrable if and only if it is $\overline{\mu}$-totally measurable, by the definitions and Remark 10-III, it follows that f_1 is $\overline{m_1}$ -totally measurable and f_2 is $\overline{m_2}$ -totally measurable. According to Theorem 2.16 (Gavriluţ and Petcu, 2007), for a real function, Gould integrability with respect to a submeasure is equivalent to its total-measurability, so f_1 is m_1 - integrable and f_2 is m_2 - integrable. Then by (Gavriluţ and Petcu, 2007), g_1 is φ_1 - integrable, g_2 is φ_2 - integrable and $\int_T g_k d\varphi_k = \int_T f_k g_k dm_k$, for $k = \overline{1,2}$. Therefore,

$$\int_T Gd\varphi = [\int_T g_1 d\varphi_1, \int_T g_2 d\varphi_2] = [\int_T f_1 g_1 dm_1, \int_T f_2 g_2 dm_2] = \int_T FGd\mu.$$

We now provide a theorem of measure change type. With this end in mind, suppose $T, T^{'}$ are two non-empty sets, $G : T \to T^{'}$ is a bijective function, $\mathcal{A}$ is an algebra of subsets of T and $\mu : \mathcal{A} \to \mathcal{R}_+$ is a set function with $\mu(A) = 0$.

Then $\mathcal{A}^{'} = \{A^{'} \subseteq T^{'}; G^{-1}(A^{'}) \in \mathcal{A}\}$ is an algebra of subsets of $T^{'}$ and we can define $\mu^G : \mathcal{A}^{'} \to \mathcal{R}_+$ by $\mu^G(A^{'}) = [m_1^G(A^{'}), m_2^G(A^{'})]$, for every $A^{'} \in \mathcal{A}^{'}$, where $m_i^G : \mathcal{A}^{'} \to \mathcal{R}_+$, $m_i^G = m_i(G^{-1}(A^{'})), \forall A^{'} \in \mathcal{A}^{'}$.

Theorem 22. *Let* $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathcal{R}_+)$ *be a q-multisubmeasure of finite variation. If* $F : T^{'} \to \mathcal{P}_{kc}(\mathcal{R}_+)$ *is* μ^G *-integrable on* $T^{'}$ *, then* $FoG : T \to \mathcal{P}_{kc}(\mathcal{R}_+)$*, defined by*

$$(FoG)(t) = [(f_1 oG)(t), (f_2 oG)(t)], \forall t \in T,$$

is μ -integrable on T and, moreover,

$$\int_{T^{'}} Fd\mu^G = \int_T FoGd\mu.$$

Proof. One can easily check that if μ is a q-multisubmeasure of finite variation, then μ^G is a q-multisubmeasure of finite variation too. Now the conclusion follows by Proposition 18 and Theorem 2.13 (Gavriluţ and Petcu, 2007).

In the following we shall obtain some results concerning integrability on atoms.

We recall that Theorem 2.16 (Gavriluţ and Petcu, 2007) shows that totally-measurability in variation and Gould integrability are equivalent on any subset of $A \in \mathcal{A}$, when dealing with submeasures $m : \mathcal{A} \to \mathcal{R}_+$. Theorem 19 proves that a similar result holds when dealing with q-multisubmeasures. In the next result we prove that this equivalence remains valid on atoms in weaker hypothesis, *i.e.*, when μ is only null-additive and q-monotone.

Theorem 23. *Suppose* $\mu : \mathcal{A} \to \mathcal{P}_{kc}(\mathcal{R}_+)$ *is null-additive, q-monotone, of finite variation and* $A \in \mathcal{A}$ *is a q-atom of* μ. *Then* F *is* μ*-integrable on* A *if and only if* F *is* $\overline{\mu}$ *- totally measurable on* A.

Proof. According to Remark 15-II, F is $\overline{\mu}$ -totally measurable if and only if f_1, f_2 are $\overline{m_2}$ -totally measurable. Since $m_1 \le m_2$ it results that f_1 is also $\overline{m_1}$ -totally measurable. Now the statement follows by Corollary 3.7 (Gavriluţ, 2011) and Proposition 18.

In what follows let T be a locally compact Hausdorff topological space, $\mathcal{K}$ the lattice of all compact subsets of T, $\mathcal{B}$ the Borel σ-algebra (*i.e.*, the smallest σ-algebra containing $\mathcal{K}$) and τ the class of all open sets.

Definition 24. *$\mu:\mathcal{B}\to\mathcal{P}_{kc}(\mathbb{R}_+)$ is said to be regular if for each set $A\in\mathcal{B}$ and each $\varepsilon>0$, there exist $K\in\mathcal{K}$ and $D\in\tau$ such that $K\subseteq A\subseteq D$ and $|\mu(D\setminus K)|<\varepsilon$.*

Remark 25. We see that μ is regular if and only if the same is m_2 in the sense of Pap (1995).

Theorem 26. *Let $\mu:\mathcal{B}\to\mathcal{P}_{kc}(\mathbb{R}_+)$ be a regular q-multisubmeasure. If $A\in\mathcal{B}$ is a q-atom of μ, there exists a unique point $a\in A$ such that $\mu(A)=\mu(\{a\})$ and $\mu(A\setminus\{a\})=\{0\}$.*

Proof. Since μ is q-multisubmeasure, then μ is null-additive, so m_1 and m_2 are null-additive. One can easily check that m_1 and m_2 are also monotone and regular (in the sense of (Pap, 1995)). Let $A\in\mathcal{B}$ be a q-atom of μ. Then A is an atom of both m_1 and m_2. Applying Theorem 9.6 (Pap, 1995) for m_1 and m_2, respectively there exist unique points $a_1\in A$ for m_1 and $a_2\in A$ for m_2, respectively such that $m_1(A)=m_1(\{a_1\})$, $m_1(A\setminus\{a_1\})=0$ and $m_2(A)=m_2(\{a_2\})$, $m_2(A\setminus\{a_2\})=0$, respectively.

We shall prove that $a_1=a_2$. Suppose this is not true, *i.e.*, $a_1\neq a_2$. Since $a_1,a_2\in A$, then $\{a_1\}\subseteq A\setminus\{a_2\}$. By the monotonicity of m_2 we have $m_2(\{a_1\})\le m_2(A\setminus\{a_2\})=0$. Hence, $m_2(\{a_1\})=0$. Since $m_1\le m_2$ it follows that $m_1(\{a_1\})=0$, which contradicts the fact that $m_1(\{a_1\})=m_1(A)>0$. Therefore, there is only one point $a\in A$ such that $m_i(A)=m_i(\{a\})$, $m_i(A\setminus\{a\})=0$, for $i=\overline{1,2}$.

Theorem 27. *Suppose $\mu:\mathcal{B}\to\mathcal{P}_{kc}(\mathbb{R}_+)$ is a regular q-multisubmeasure and $F:T\to\mathcal{P}_{kc}(\mathbb{R}_+)$. If $A\in\mathcal{B}$ is a q-atom, then F is μ-integrable on A and*

$$\int_A F d\mu = F(a)\mu(\{a\}),$$

where $a\in A$ is the single point resulting by the previous theorem.

Proof. According to Theorem 19, we prove that F is $\overline{\mu}$ -totally measurable. Indeed, by Theorem 26 there exists a unique point $a \in A$ so that $\mu(A\setminus\{a\}) = \{0\}$. Then the partition $P = \{A\setminus\{a\},\{a\}\}$ assures the $\overline{\mu}$ -totally measurability of F on A. Applying now Proposition 18 and Theorem 4.4 (Candeloro *et al*., submitted for publication) we obtain the equality.

Corollary 28. *If* T *is a compact Hausdorff space,* $\mu : \mathcal{B} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ *is a regular finitely purely atomic, q-multisubmeasure (where* $T = \bigcup_{i=1}^{p} A_i$ *and* $A_i \in \mathcal{B}, i = \overline{1, p}$ *are pairwise disjoint q-atoms of* μ*) and if* $F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$ *is* μ *-integrable on* T *, then for every* $i = \overline{1, p}$ *, there exist unique points* $a_i \in A_i$ *so that* $\mu(A_i \setminus\{a_i\}) = \{0\}$ *and, in this case,*

$$\int_T F d\mu = F(a_1)\mu(\{a_1\}) + \cdots + F(a_p)\mu(\{a_p\}).$$

Corollary 29. (*Lebesgue Type Theorem*) *Suppose* $\mu : \mathcal{B} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ *is a regular q-multisubmeasure. If for every* $n \in \mathbb{N}$*,* $F_n, F : T \to \mathcal{P}_{kc}(\mathbb{R}_+)$*, where* $F_n = [f_n^1, f_n^2]$ *are* μ *-integrable on an atom* $A \in \mathcal{B}$ *of* μ *and if with respect to* h*,* $(F_n)_n$ *pointwise converges to* F *, then*

$$\lim_{n\to\infty} \int_A F_n d\mu = \int_A F d\mu.$$

Proof. By virtue of Theorems 26 and 27, there exists a unique point $a \in A$ such that $\mu(A\setminus\{a\}) = \{0\}$ and for every $n \in \mathbb{N}$, $\int_A F_n d\mu = F_n(a)\mu(A)$ and $\int_A F d\mu = F(a)\mu(A)$. Since $(F_n)_n$ pointwise converges to F with respect to h,

$$h(\int_A F_n d\mu, \int_A F d\mu) = h(F_n(a)\mu(A), F(a)\mu(A)) \le h(F_n(a), F(a))\overline{\mu}(A) \to 0.$$

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INTEGRABILITATE ÎN CAZUL MULTIFUNCŢIILOR (DE MULŢIME) CU VALORI INTERVAL

(Rezumat)

Recent, am propus un nou tip de integrală a unei multifuncţii cu valori interval în raport cu o multifuncţie de mulţime cu valori interval. În această lucrare, continuăm studiul acestui tip de integrală, stabilind diferite proprietăţi specifice ale acesteia. De asemenea, discutăm unele proprietăţi referitoare la integrabilitatea de tip Gould pe atomi.