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SELF-MODULATION OF A HOLLOW CATHODE DISCHARGE PLASMA DYNAMICS
II. THEORETICAL MODELING

BY

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Abstract. A theoretical model is proposed, in the frame of Scale Relativity Theory, able to explain the phenomenon of self-modulation of a hollow cathode discharge plasma dynamics. In this model, the complexity of the interactions in the plasma volume was replaced by non-differentiability (fractality). Discharge plasma particles move free, without any constrains, on continuous but non-differentiable curves in a fractal space. A Riccati type differential equation was obtained, describing the dynamics of a harmonic oscillator. The solution of this equation shows a frequency modulation through a Stoler transformation. The obtained results are in good agreement with the experimental ones.

Keywords: non-differentiability; Scale Relativity Theory; fractal; self-modulation.

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1. Introduction

Plasma discharges can be assimilated to complex systems taking into account their structural-functional duality (Mitchell, 2009). The standard models (fluid model, kinetic model, etc.) (Morozov, 2012; Chen, 2016) used to study the plasma discharges dynamics are based on the hypothesis of differentiability of the physical variables that describe it, such as energy, momentum, density, etc. But differential methods fail when facing the physical reality, such as instabilities of the discharge plasma that can generate chaos or patterns through self-structuring, by means of the non-differentiable (fractal) method (Mandelbrot, 1982; Hastings and Sugihara, 1993; Falconer, 2014).

In order to describe some of the dynamics of plasma discharges by means of non-differentiable method, and still remain treatable as differential method, it is necessary to introduce the scale resolution, both in the expressions of the physical variables and the dynamics equations. This means that any dynamic variable become dependent also on the scale resolution. Such a physical theory was developed both in the Scale Relativity Theory with fractal dimension equals with 2 (Nottale, 1993; Nottale, 2011) and with an arbitrary constant fractal dimension (Dimitriu *et al.*, 2015; Merches and Agop, 2016). In the field of plasma discharges, if we assume that the complexity of interactions in the plasma volume is replaced by non-differentiability (fractality), the constrained motions on continuous and differentiable curves in a Euclidian space of the plasma discharge particles are replaced with the free motions, without any constrains, on continuous but non-differentiable curves in a fractal space of the same discharge plasma particles. This is the reasoning by which, at time resolution scales large by comparing with the inverse of the highest Lyapunov exponent, the deterministic trajectories are replaced by a collection of potential states, so that the concept of "definite position" is substituted by that of an ensemble of positions having a definite probability density. As a consequence, the determinism and the potentiality (non-determinism) become distinct parts of the same "evolution" of discharge plasma, through reciprocal interactions and conditioning, in such a way that the plasma discharge particles are substituted with the geodesics themselves (Arnold, 1989; Hillborn, 2000).

In the present paper, a non-differentiable theoretical model is developed, able to explain the phenomenon of self-modulation of a plasma dynamics, experimentally observed in a hollow cathode discharge in connection with the development of two space charge structures.

2. Theoretical Model and Discussion

In the frame of Scale Relativity Theory with an arbitrary constant fractal dimension, the dynamics of discharge plasma can be described by means of the covariant derivatives (Nottale, 2011):

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^l \partial_l + \frac{1}{4}(dt)^{\frac{2}{D_F}-1} D^{lk} \partial_l \partial_k, \tag{1}$$

where

$$\begin{gathered}
\partial_t = \frac{\partial}{\partial t}, \quad \partial_l = \frac{\partial}{\partial X^l}, \quad \partial_l \partial_k = \frac{\partial^2}{\partial X^l \partial X^k} \\
D^{lk} = \left(\lambda_+^l \lambda_+^k \pm \lambda_-^l \lambda_-^k\right) - i\left(\lambda_+^l \lambda_+^k \pm \lambda_-^l \lambda_-^k\right). \\
\hat{V}^l = V^l - iU^l, \quad i = \sqrt{-1}
\end{gathered} \tag{2}$$

In the above relations X^l are the spatial fractal coordinates, t is the non-fractal time coordinate, having the role of motion curve affine parameter, dt is the resolution scale, $\hat{V}^l$ is the velocity complex field, V^l is the differentiable component of the velocity complex field, which is independent on the resolution scale, U^l is the non-differentiable component of the velocity complex field, which is dependent on the resolution scale, D^{lk} is the fractal – non-fractal transition pseudo-tensor, dependent, through stochastic fractalization, either on the "forward physical processes" λ_+^l, or the "backward physical processes" λ_-^l, D_F is the fractal dimension of the motion curves. For D_F one can choose different definitions, *i.e.* the fractal dimension in a Kolmogorov sense, Hausdorff-Besikovici sense, etc. (Mandelbrot, 1982; Barnsley, 1993), but once chosen a definition, it has to remain constant during the whole analysis of the discharge plasma dynamics.

For fractalization through Markov type stochastic processes, *i.e.* for Levy type movement of the discharge plasma particles (Mandelbrot, 1982; Barnsley, 1993), the fractal – non-fractal transition pseudo-tensor becomes

$$D^{lk} = \pm 4i\lambda\delta^{lk}, \tag{3}$$

where λ is the "diffusion coefficient" associated to the fractal – non-fractal transitions (Merches and Agop, 2016) and δ^{lk} is the Kronecker pseudo-tensor. In this case, the scale covariant derivative (1) takes the form

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^l \partial_l \pm i\lambda(dt)^{\frac{2}{D_F}-1} \partial_l \partial^l. \tag{4}$$

Postulating now the scale covariance principle, according to which the physics laws in their simplest representation are remaining invariant with respect to the scale transformations, the states' density conservation law becomes

$$\frac{\hat{d}\rho}{dt}=\partial_t\rho+\hat{V}^l\partial_l\rho\pm i\lambda\left(dt\right)^{\frac{2}{D_F}-1}\partial_l\partial^l\rho=0\,, \tag{5}$$

or more, separating the movement on the scale resolution

$$\partial_t\rho+V^l\partial_l\rho=0 \tag{6}$$

for the differentiable scale resolution and

$$-U^l\partial_l\rho\pm\lambda\left(dt\right)^{\frac{2}{D_F}-1}\partial_l\partial^l\rho=0 \tag{7}$$

for the non-differentiable scale resolution. From such a perspective, the fractal – non-fractal dynamic transition of the states' density can be obtained by summing Eqs. (6) and (7), taking the form:

$$\partial_t\rho+\left(V^l-U^l\right)\partial_l\rho\pm\lambda\left(dt\right)^{\frac{2}{D_F}-1}\partial_l\partial^l\rho=0\,. \tag{8}$$

From here, by means of compactification of the movements at the two scale resolutions $V^l = U^l$, the fractal type diffusion equation become:

$$\partial_t\rho-\lambda\left(dt\right)^{\frac{2}{D_F}-1}\partial_l\partial^l\rho=0\,. \tag{9}$$

Let us now use Eq. (9) to analyze the dynamic of an electron beam accelerated in a strong electric field which impinges onto a neutral medium. As a result of these interactions, ionizations are produced both by the primary electrons (from the beam), αj, where α is the primary ionization coefficient and j is the beam current density, and by the secondary electrons which result from the direct ionization processes, $\beta j\rho_e$, with β the secondary ionization coefficient and ρ_e the electron density. In these conditions, the focus is placed on the study of the dynamics induced only by the electronic branch, through Eq. (9) written in the following form:

$$\partial_t\rho_e+\lambda\left(dt\right)^{\frac{2}{D_F}-1}\partial_l\partial^l\rho_e=-\alpha j-\beta j\rho_e\,. \tag{10}$$

Since the previous dynamics implied a one-dimensional symmetry, Eq. (10), by means of substitutions

$$\alpha j+\beta j\rho_e=Kq,\quad \tau=\frac{x}{v}-t,\quad M=\frac{K}{\beta jv^2}\lambda\left(dt\right)^{\frac{2}{D_F}-1},\quad 2Rw=-\frac{K}{\beta j}, \tag{11}$$

becomes a damped harmonic oscillator type equation:

$$M\ddot{q} + 2R\dot{q} + Kq = 0\,. \tag{12}$$

Rewritten as

$$\begin{aligned}\dot{p} &= -\frac{2R}{M}p - \frac{K}{M}q\\ \dot{q} &= p\end{aligned}\,, \tag{13}$$

Eq. (12) induces a two-dimensional manifold of phase space type (p,q), in which p would corresponds to a "momentum" type variable and q to a "position" type one. Then, the parameters M, R and K can have the following significance:

i) M represents the "matricidal" type effects through the connection with ionization processes (both global, described by $\alpha j + \beta j$, and local, described by βj) and through the fractal diffusion (described by $\lambda\left(dt\right)^{\frac{2}{D_F}-1}$). All these are done with respect to a travelling wave type movement based on the self-similar dynamic solutions ($\tau = \frac{x}{v} - t$);

ii) R represents the "dissipative" type effects through the connection with the ionization processes (both global, described by $\alpha j + \beta j$, and local, described by βj);

iii) K represents the "structural" type effects in connection with the ionization processes (only the global ones, described by $\alpha j + \beta j$).

The second equation from (13) corresponds to the momentum definition. Eqs. (13) do not represents a Hamiltonian system, since the associated matrix is not an involution (the matrix trace is not null). This statement becomes clearer if we put the system in its matrix form:

$$\begin{pmatrix}\dot{p}\\ \dot{q}\end{pmatrix} = \begin{pmatrix}-2\dfrac{R}{M} & -\dfrac{K}{M}\\ 1 & 0\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}. \tag{14}$$

As long as M, R and K have constant values, this matrix equation written in the equivalent form evidences the position of the energy and thus of the Hamiltonian, which is, for this particular case, identified with the energy of the system obviously only for the cases in which the energy can be identified with the Hamiltonian. Indeed, from Eq. (14) it can be obtained

$$\frac{1}{2}M\left(p\dot{q}-q\dot{p}\right)=\frac{1}{2}\left(Mp^2+2Rpq+Kq^2\right), \tag{15}$$

which proves that the energy in its quadratic form (the right hand of Eq. (15)) is the variation rate of the physical action, represented by the elementary area from the phase space. From here it results that the energy does not have to obey the conservation laws in order to act like a variation rate for the physical action.

On can ask now what could be the conservation law, if it exists. To give an adequate answer, we first observe that Eq. (15) can be written as a Riccati type differential equation

$$\dot{w}+w^2+2\mu w+w_0^2=0, \tag{16}$$

with

$$w=\frac{p}{q},\quad \mu=\frac{R}{M},\quad \omega_0^2=\frac{K}{M}. \tag{17}$$

Furthermore let us note that Riccati type Eq. (16) always represents a Hamiltonian system describing harmonic oscillator type dynamic

$$\begin{pmatrix}\dot{p}\\ \dot{q}\end{pmatrix}=\begin{pmatrix}-\frac{R}{M} & -\frac{K}{M}\\ 1 & \frac{R}{M}\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}. \tag{18}$$

This is a general characteristic describing the Riccati type equation and the Hamiltonian's dynamic (Arnold, 1989; Libermann and Marle, 1987). Eq. (9) can be reobtained by bulding from Eq. (18) the 1- differential form for the elementary area from the phase space for harmonic oscillator type dynamic. Regarding Eq. (15), it can be integrated by specifying the fact that the energy does not conserve anymore, but we find that another more complicated dynamics variable will be conserved (Denman, 1968):

$$\frac{1}{2}\left(Mp^2+2Rpq+Kq^2\right)\exp\left[\frac{2R}{\sqrt{MK-R^2}}\arctan\left(\frac{Mp+Rq}{q\sqrt{MK-R^2}}\right)\right]=\text{const}. \tag{19}$$

It results that the energy is conserved in a classical meaning when either R becomes null, or the movement in the phase space is characterized by the line passing through origin and having the slope defined by the ratio between R and

M. Moreover, by comparing with the case of thermal radiation regarding the distribution function on a pre-established "local oscillators" ensemble, it results

$$P(r,w)=\frac{1}{1+2rw+w^2}\exp\left\{\frac{2r}{\left(1-r^2\right)^{1/2}}\arctan\left[\frac{w\left(1-r^2\right)^{1/2}}{1-rw}\right]\right\}, \tag{20}$$

where r is the correlation coefficient and $w^2=\frac{\varepsilon_0}{u}$ is the ratio between the thermal energy quanta, ε_0, and the reference energy, u. Eq. (19) can be rewritten as:

$$\frac{Kq^2}{2}=\frac{\text{const}}{1+2rw+w^2}\exp\left\{\frac{2r}{\left(1-r^2\right)^{1/2}}\arctan\left[\frac{w\left(1-r^2\right)^{1/2}}{1-rw}\right]\right\}, \tag{21}$$

with

$$w^2=\frac{Mp^2}{Kq^2},\quad r^2=\frac{R}{K}. \tag{22}$$

From here we can emphasis the statistic character of the energy: the potential energy is constructed as a functional of a specific statistical variable. This variable is given by the ratio between the kinetic energy and the potential one of the local oscillator.

Thus, we propose here such a "quantization" procedure (see Fig. 1a and 1b) through the correlation of all statistical ensembles associated with "local oscillators" (Ioannidou, 1983), induced by mean of the condition

$$P(r,1)=\frac{e^{-\frac{\varepsilon_0}{kT}}}{2(1+r)}\exp\left[\frac{2r}{\left(1-r^2\right)^{1/2}}\arctan\left(\sqrt{\frac{1+r}{1-r}}\right)\right], \tag{23}$$

where k is the Boltzmann constant and T is the characteristic temperature of the thermal radiation, representing explicitly the connection between the "quanta" and the statistical correlation of the process represented by the thermal radiation. Moreover, the previous relation does not explicitly specifies the expression of the "quanta" in the weak correlation limit since as for $r\to 0$ it implies $\varepsilon_0\to kT\ln 2$. In such a limit, the quanta and thus the frequency ν (through $\varepsilon_0 = h\nu$, where h is the Planck constant) is proportional with the "color" temperature. We note that in our case the thermal radiation is identified with the thermodynamic equilibrium plasma radiation.

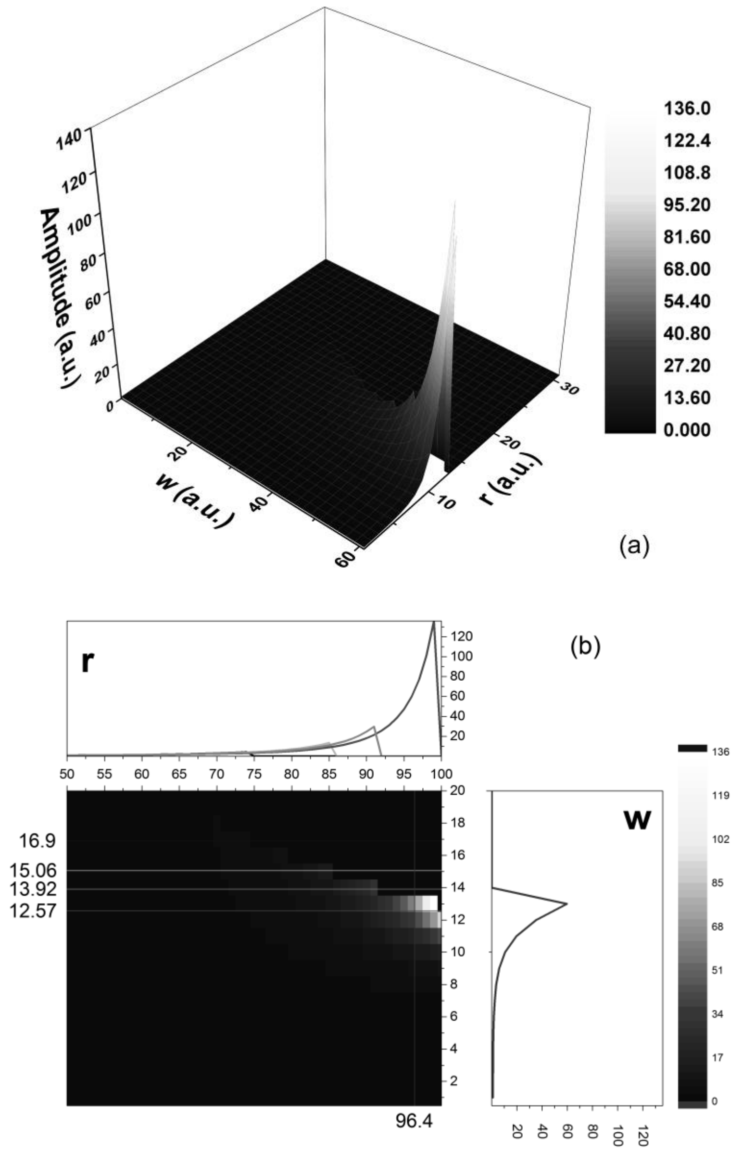

Fig. 1 – "Quantization" procedure through correlation of all statistical ensembles associated with "local oscillators": 3D dependences (*a*) and the contour plot (*b*).

Since we are focused on identifying the dissipative forces, we will present a physical significance for the Riccati Eq. (16) by means of its associated Hamiltonian system (18). Let us observe that Eq. (12) is the expression of a variation principle

$$\delta\int_{t_0}^{t_1} L dt = 0 \tag{24}$$

regarding the Lagrangian

$$L\left(q,\dot{q},t\right)=\frac{1}{2}\left(M\dot{q}^2-Kq^2\right)\exp\left(\frac{2R}{M}t\right). \tag{25}$$

This represents the Lagrangian form of a harmonic oscillator with explicit time dependent parameters. The Lagrangian integral defined on a finite interval [t_0,t_1] is the physical action of an oscillator during that specific time interval, describing the difference between the kinetic and potential energy, respectively. In order to obtain Eq. (12), it is necessary to consider the variation of this action under the explicit condition in such a way that the variance of the coordinate at the interval extremes is null:

$$\delta q\Big|_{t_0}=\delta q\Big|_{t_1}=0. \tag{26}$$

Even so, in order to obtain a closed trajectory, we need to impose a supplementary condition, for instance that the values of the coordinates at the interval extremes are identical:

$$q\left(t_0\right)=q\left(t_1\right). \tag{27}$$

Moreover, if this trajectory is closed in the phase space, it will result that the same condition will be true also for velocities.

Let us focus now on the movement principle and on the movement equation. The Lagrangian is defined until an additive function, which needs to be derivative in respect to the time of another function. The procedure is largely used in theoretical physics by defining the gauge transformation. Let's define a gauge transformation in which the Lagrangian is a perfect square. This is known and explored in the control theory (Zelkin, 2000). The procedure consists in adding the following term to Lagrangian

$$\frac{1}{2}\frac{d}{dt}\left[wq^2\exp\left(\frac{2R}{M}t\right)\right], \tag{28}$$

where w is a continuous function in time, so that the Lagrangian is a perfect square. The function variation given by the derivative operator is null, only due to the conditions presented in Eq. (27), thus the motion equation does not change. The new Lagrangian written in relevant coordinates takes the following form:

$$L\left(\dot{q},q,t\right)=\frac{M}{2}\left(\dot{q}+\frac{w}{M}q\right)^{2}\exp\left(\frac{2R}{M}t\right), \tag{29}$$

with the condition that *w* need to satisfy the following Riccati type equation:

$$\dot{w}-\frac{1}{M}w^{2}+\frac{2R}{M}w-K=0. \tag{30}$$

Lagrangian depicted in Eq. (29) will be considered here as representing the whole energy of the system. As before, there is a relationship between the Riccati Eq. (30) and the Hamiltonian dynamics. Henceforth we will find a relation similar to that one presented in Eq. (18):

$$\begin{pmatrix}\dot{\eta}\\ \dot{\xi}\end{pmatrix}=\begin{pmatrix}-\dfrac{R}{M} & -\dfrac{K}{M}\\ 1 & \dfrac{R}{M}\end{pmatrix}\begin{pmatrix}\eta\\ \xi\end{pmatrix}, \tag{31}$$

with $w=\dfrac{\eta}{\xi}$. This system is obviously a Hamiltonian one. Thus, we can identify the factors of *w* as the phase space coordinates. Eq. (30) specifies the fact that *w* is a dissipation coefficient, more precisely a mass variation rate for the variable mass case. It is important to find the most general solution of this equation. Cariñena and Ramos (Cariñena and Ramos, 2000) presented a modern approach to integrate a Riccati equation. Let's consider the next complex numbers:

$$w_0\equiv R+iM\Omega,\quad w_0^{*}\equiv R-iM\Omega,\quad \Omega^{2}=\frac{K}{M}-\left(\frac{R}{M}\right)^{2}. \tag{32}$$

The roots of the quadratic polynomial from the left hand side of Eq. (30) are two constant solution of the equation. Being constant, their derivatives are null, thus the polynomial is also null. In order to avoid this, we first perform the homograph transformation:

$$z=\frac{w-w_0}{w-w_0^{*}}. \tag{33}$$

In these conditions, it results z is a solution of the linear and homogeneous first order equation:

$$\dot{z} = 2i\Omega z \quad \Rightarrow \quad z(t) = z(0)e^{2i\Omega t}. \tag{34}$$

Hence, if we express the initial condition $z(0)$ in a right manner, we can obtain the general solution of Eq. (30) by applying an inverse transformation to Eq. (33). We find

$$w = \frac{w_0 + re^{2i\Omega(t-t_r)}w_0^*}{1 + re^{2i\Omega(t-t_r)}}, \tag{35}$$

where r and t_r are two real constants characterizing the solution. Using relations (32), we can put the same solution in real terms:

$$z = R + M\Omega\left\{\frac{2r\sin\left[2\Omega(t-t_r)\right]}{1+r^2+2r\cos\left[2\Omega(t-t_r)\right]} + i\frac{1-r^2}{1+r^2+2r\cos\left[2\Omega(t-t_r)\right]}\right\}. \tag{36}$$

This relationship shows a frequency modulation through a Stoler transformation (Stoler, 1970) which leads to the complex representation of this parameter.

Fig. 2 shows the dimensionless discharge current oscillations, obtained from the solution (36) for different scale resolutions of the frequency, r being kept constant at the value 0.1. We observe that for small scale resolutions the current is described by a simple oscillatory regime, while as the frequency scale resolution increases we notice the appearance of some patterns. The patterns become denser and foreshadow the presence of modulation of the oscillating frequency.

From Fig. 2 we can extract time series of the discharge current oscillations for different value of ω, which are shown in Fig. 3. We notice that these signals are similar to that experimentally recorded.

Fig. 4 shows the time series of the discharge current oscillations for different values of r and for two values of the oscillations frequency, ω. The damping of the oscillatory state describes the losses through dissipative or dispersive mechanisms. In Fig. 4 competing oscillatory behaviors described by two oscillations frequencies, with comparable amplitudes, can be identified. As the damping increases, the ratio between the two oscillation frequencies changes, the system ending in an oscillatory state on a single frequency. These results are also in good agreement with the experimental ones.

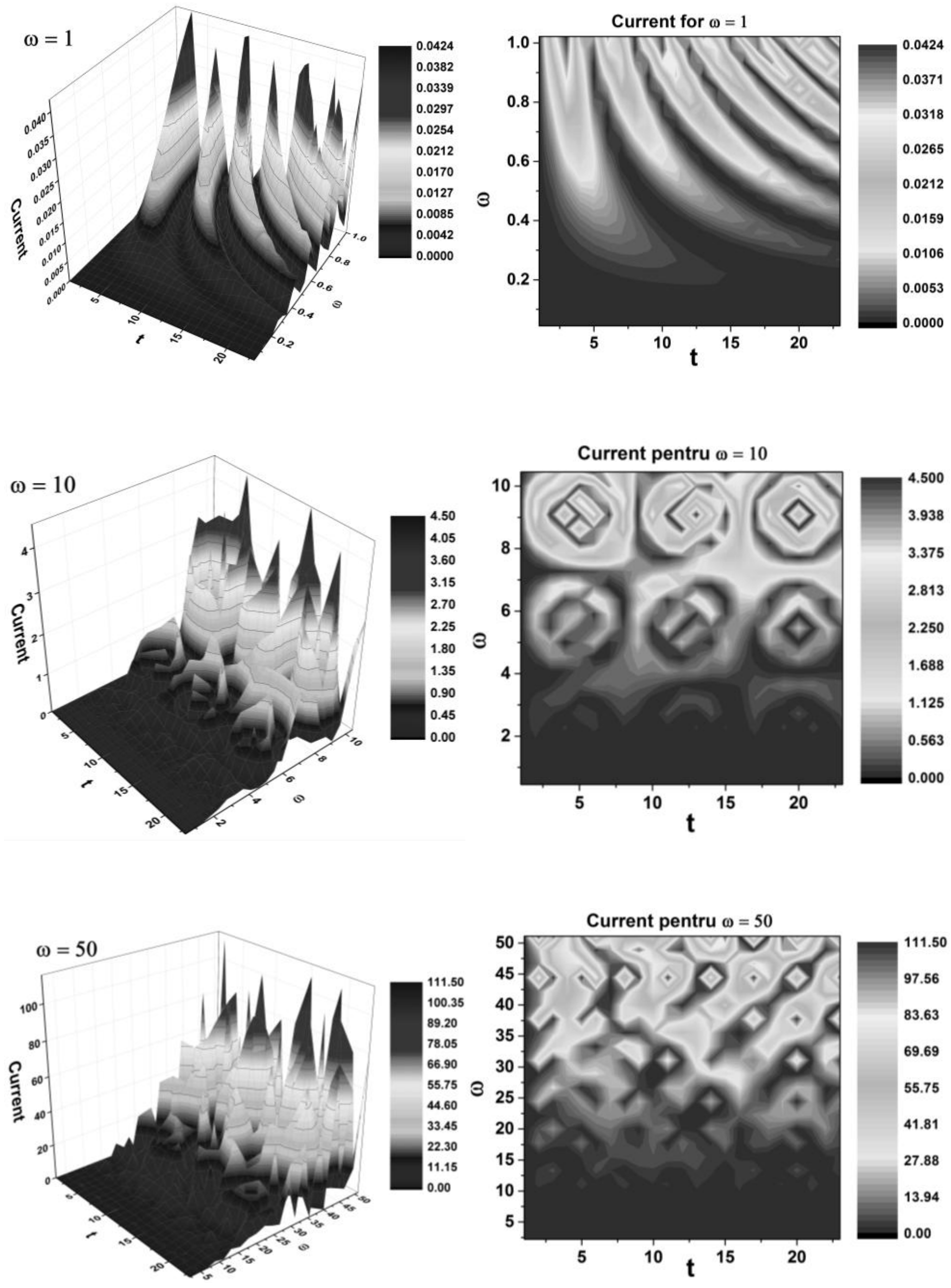

Fig. 2 – Dimensionless discharge current obtained from the theoretical model, for different scale resolutions of the oscillations frequency (3D maps on the left column and the contour plots on the right column, respectively).

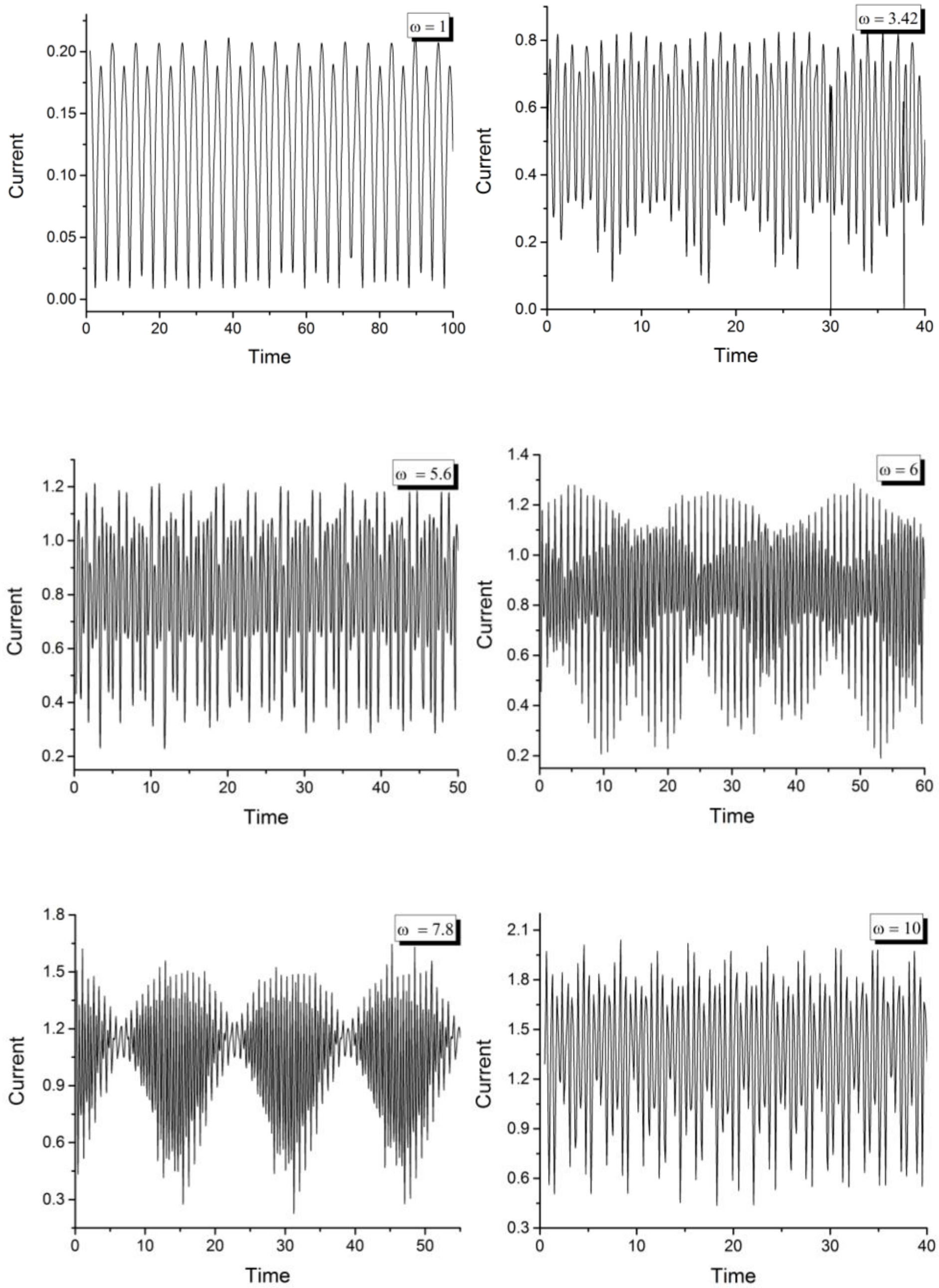

Fig. 3 – Time series of the discharge current obtained from the theoretical model, for different value of ω.

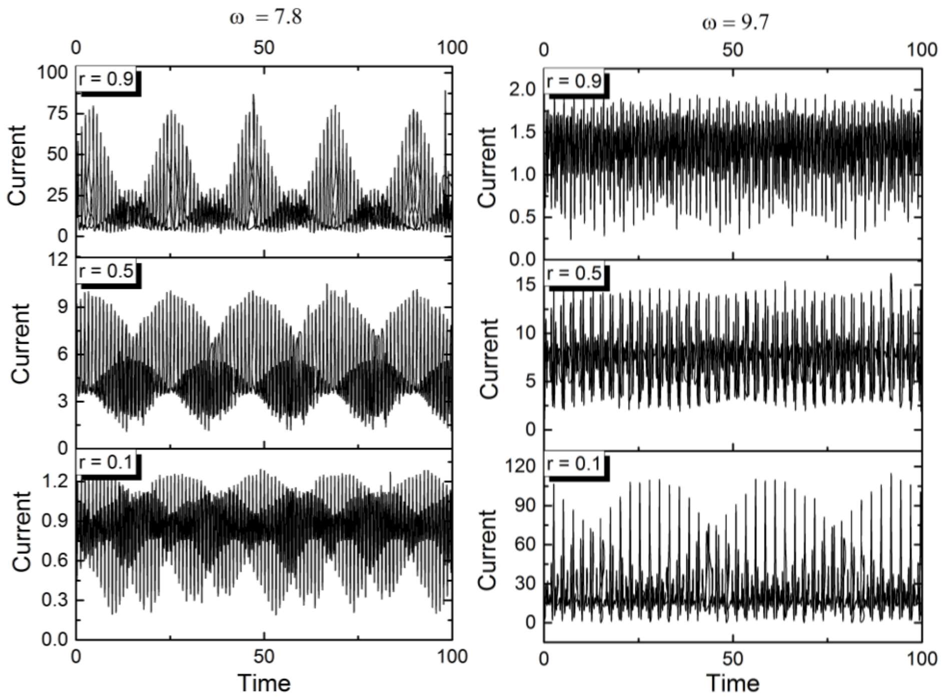

Fig. 4 – Time series of the discharge current obtained from the theoretical model, for different value of *r* and two values of ω.

3. Conclusions

By assuming that the discharge plasma particles moves on continuous but non-differentiable (fractal) curves, a theoretical model was developed in the frame of Scale Relativity Theory, able to explain the phenomenon of self-modulation of the plasma system dynamics. The obtained results from the this theoretical model are in good agreement with the experimentally recorded ones.

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AUTOMODULAREA DINAMICII UNEI PLASME DE DESCĂRCARE CU CATOD CAVITAR
II. Modelare teoretică

(Rezumat)

Este propus un model teoretic, dezvoltat în cadrul Teoriei Relativității de Scală, capabil să explice automodularea dinamicii unei plasme de descărcare cu catod cavitar. În cadrul acestui model, complexitatea interacțiunilor din volumul de plasmă a fost înlocuită de nediferențiabilitate (fractalitate). Particulele din plasma de descărcare se mișcă liber, fără constrângeri, pe curbe continue dar nediferențiabile, într-un spațiu fractal. S-a obținut o ecuație de tip Riccati, ce descrie dinamica unui oscilator armonic. Soluția acestei ecuații prezintă o modulare a frecvenței prin intermediul unei transformări Stoler. Rezultatele obținute sunt în bună concordanță cu cele experimentale.