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GENERALIZATION OF NABLA OPERATOR IN FRACTIONAL ONE DIMENSIONAL SPACE

BY

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Abstract. Nabla operator in fractional one dimensional space have to be used for modeling the various dissipative systems, as the application in this paper proves.

Keywords: Fractional dimensional space; Nabla operator; dissipative systems.

1. Introduction

The non-integer dimension (D) occurs in certain key quantities such as Gaussian integral:

$$\int e^{-\alpha r^2} d\mathbf{r} = \left(\frac{\pi}{\alpha}\right)^{D/2}$$

Or the radial Laplace operator

$$\frac{\partial^2}{\partial r^2} + \frac{(D-1)}{r}\frac{\partial}{\partial r}$$

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This paper presents a mathematically concrete realisation of spaces with non-integer *D*.

2. Nabla Operator in Fractional Space

In the 1970s, (Stillinger, 1977) generalized the Laplace operator in a fractional space of dimension *D*, where *D* is a non-integer number:

$$\nabla_D^2 f(r) = f''(r) + \frac{D-1}{r} f'(r) \text{ , } 0 < D \le 1 \tag{1}$$

This confirm the compatibility of expressions in the introduction. We can generalize for 3 orthogonal coordinates:

$$\nabla_D^2 f(r) = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z} \tag{2}$$

where $0 < \alpha_1 \le 1,\ 0 < \alpha_2 \le 1,\ 0 < \alpha_3 \le 1$ and $\alpha_1 + \alpha_2 + \alpha_3 = 3$

For single variable Laplacian operator in fractional space:

$$\nabla_D^2 f(r) = \frac{\partial^2}{\partial x^2} + \frac{\alpha - 1}{x}\frac{\partial}{\partial x} = |\nabla_\alpha| \mathbf{u}_x \tag{3}$$

Using binomial series, ignoring terms involving second or higher degree of *x*, we obtain:

$$\nabla_\alpha = \left(\frac{d}{dx} + \frac{1}{2}\frac{\alpha - 1}{x}\right)\mathbf{u}_x \text{ , } 0 < \alpha \le 1 \tag{4}$$

where $\mathbf{u}_x$ is the unitary vector of x direction.

3. Application

Let consider the Newton’s low in one-dimensional fractional space:

$$\frac{d^2 x(t)}{dt^2} = -\frac{1}{m}\nabla_\alpha U(x) \tag{5}$$

where m is a mass and $U(x)$ is a potential function. Let as an example, m = 1 and the potential function

$$U(x) = -x + 1 \tag{6}$$

In this condition, the Eq. (5) become:

$$\frac{d^2 x(t)}{dt^2} = -\left(\frac{d}{dx} + \frac{1}{2}\frac{\alpha - 1}{x}\right)(-x + 1) \tag{7}$$

After some algebraic manipulations we obtain:

$$x(t)x''(t) - \frac{\alpha + 1}{2}x(t) + \frac{\alpha - 1}{2} = 0 \tag{8}$$

Solving this equation using MATHEMATICA 8, for initial conditions

$$x(0) = 0.1\ ,\ x'(0) = 0 \tag{9}$$

we can obtain the solutions for different values of α. In Fig.1 we present these solution in the form of speed, for three different values of $\alpha = 0.3, 0.7, 1$.

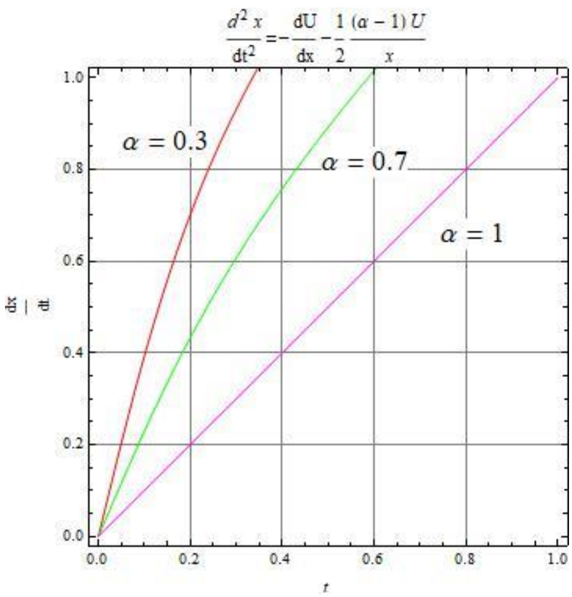

Fig. 1 – The dependence of the speed $x'(t)$ for three different values of fractal dimension α.

3. Conclusions

The main conclusion of the present paper a the following:

i) The expression of the fractional Nabla operator in three dimensional space was given.

ii) In the one dimensional space, the speed dependence of time, for various fractal dimension is obtained. The standard (*i.e.* non-fractal) case, corresponding to the uniform accelerated motion, is obtained for α = 1. For $\alpha < 1$, which represent the fractal case, we have the dissipative systems.

REFERENCES

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GENERALIZAREA OPERATORULUI NABLA ÎNTR-UN SPAȚIU UNIDIMENSIONAL FRACȚIONAL

(Rezumat)

Operatorul Nabla definit pentru un spațiu unidimensional, de dimensiune fracționară, poate fi folosit pentru a modela diverse sisteme cu pierderi, așa cum se demonstrează în aplicația din lucrare.