BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Volumul 63 (67), Numărul 4, 2017
Secţia
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

EXISTENCE OF POSITIVE SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM

BY

RODICA LUCA*

“Gheorghe Asachi” Technical University of Iaşi,
Department of Mathematics

Received: November 25, 2017
Accepted for publication: December 10, 2017

Abstract. We study the existence and multiplicity of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with nonnegative and nonsingular nonlinearities, subject to multi-point boundary conditions which contain fractional derivatives.

Keywords: Riemann-Liouville fractional differential equations; multi-point boundary conditions; positive solutions; existence; multiplicity.

1. Introduction

Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (for example, the primary infection with HIV), economics, control theory, signal and image processing, thermoelasticity, aerodynamics, viscoelasticity, electromagnetics and rheology (Arafa *et al*., 2012; Baleanu *et al*., 2012; Cole, 1993; Das, 2008; Ding and Ye, 2009; Djordjevic *et al*., 2003; Ge and Ou, 2008; Kilbas *et al*., 2006; Klafter *et al*., 2011; Metzler and Klafter, 2000; Ostoja-Starzewski, 2007; Podlubny, 1999; Povstenko, 2015; Sabatier *et al*., 2007; Samko *et al*., 1993; Sokolov *et al*., 2002). Fractional differential

*Corresponding author; *e-mail*: rluca@math.tuiasi.ro

equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations.

We consider the system of nonlinear ordinary fractional differential equations

$$(S)\qquad \begin{cases} D_{0+}^{\alpha}u(t)+f\big(t,u(t),v(t)\big)=0, & t\in(0,1),\\ D_{0+}^{\beta}v(t)+g\big(t,u(t),v(t)\big)=0, & t\in(0,1),\end{cases}$$

with the multi-point boundary conditions

$$(BC)\quad \begin{cases} u^{(j)}(0)=0,\ \ j=0,\dots,n-2;\ \ D_{0+}^{p_1}u(t)|_{t=1}=\sum_{i=1}^{N} a_i D_{0+}^{q_1}u(t)|_{t=\xi_i}\,,\\ v^{(j)}(0)=0,\ \ j=0,\dots,m-2;\ \ D_{0+}^{p_2}v(t)|_{t=1}=\sum_{i=1}^{M} b_i D_{0+}^{q_2}v(t)|_{t=\eta_i}\,,\end{cases}$$

where $\alpha,\beta\in\mathbb{R},\ \alpha\in(n-1,n],\ \beta\in(m-1,m],\ n,m\in\mathbb{N},\ n,m\geq 3, p_1,p_2,$ $q_1,q_2\in\mathbb{R}, p_1\in[1,n-2], p_2\in[1,m-2],\ q_1\in[0,p_1], q_2\in[0,p_2],\ \xi_i,a_i\in$ $\mathbb{R}$ for all $i=1,\dots,N\ (N\in\mathbb{N}),\quad 0<\xi_1<\cdots<\xi_N\leq 1, \eta_i,b_i\in\mathbb{R}$ for all $i=1,\dots,M\ (M\in\mathbb{N}), 0<\eta_1<\cdots<\eta_M\leq 1$, and D_{0+}^{k} denotes the Riemann-Liouville derivative of order k (for $k=\alpha,\beta,p_1,p_2,q_1,q_2$).

Under sufficient conditions on the nonnegative and nonsingular functions f and g, we study the existence and multiplicity of positive solutions of problem (S)-(BC). We use some theorems from the fixed point index theory (Amann, 1976; Zhou and Xu, 2006). By a positive solution of problem (S)-(BC) we mean a pair of functions $(u,v)\in(C([0,1],[0,\infty)))^2$ satisfying (S) and (BC) with $u(t)>0$ for all $t\in(0,1]$ or $v(t)>0$ for all $t\in(0,1]$.

The system (S) with some positive parameters, subject to the boundary conditions (BC) was investigated in (Henderson *et al*., 2017). The system (S) with $f(t,u,v)=\tilde{f}(t,v),\ g(t,u,v)=\tilde{g}(t,u)$ has been studied in (Henderson and Luca, 2017c). In this last paper, the authors use some different operators and different assumptions than those we use in this paper. The existence of positive solutions of the system (S) with the coupled multi-point boundary conditions

$$(\widetilde{BC})\quad \begin{cases} u^{(j)}(0)=0,\ \ j=0,\dots,n-2;\ \ D_{0+}^{p_1}u(t)|_{t=1}=\sum_{i=1}^{N} a_i D_{0+}^{q_1}v(t)|_{t=\xi_i}\,,\\ v^{(j)}(0)=0,\ \ j=0,\dots,m-2;\ \ D_{0+}^{p_2}v(t)|_{t=1}=\sum_{i=1}^{M} b_i D_{0+}^{q_2}u(t)|_{t=\eta_i}\,,\end{cases}$$

was studied in (Henderson and Luca, 2017b). For other papers which investigate the existence, nonexistence and multiplicity of positive solutions for systems of fractional differential equations with nonnegative or sign-changing nonlinearities, subject to various nonlocal boundary conditions we mention (Henderson and Luca, 2014a, b; Luca and Tudorache, 2014; Henderson and Luca, 2015; Henderson *et al*., 2015; Henderson and Luca, 2016a, b).

The paper is organized as follows. In Section 2, we present some auxiliary results which investigate a nonlocal boundary value problem for fractional differential equations, and we give the properties of the Green functions associated to our problem. Section 3 contains the existence and multiplicity results for the positive solutions of problem (S)-(BC).

2. Auxiliary Results

We present here some auxiliary results from (Henderson and Luca, 2017a) that will be used to prove our main results.

We consider the fractional differential equation

$$D_{0+}^{\alpha}u(t) + x(t) = 0, \quad 0 < t < 1, \tag{1}$$

with the multi-point boundary conditions

$$u^{(j)}(0) = 0, \; j = 0, \dots, n-2; \; D_{0+}^{p_1}u(t)|_{t=1} = \sum_{i=1}^{N} a_i D_{0+}^{q_1}u(t)|_{t=\xi_i}, \tag{2}$$

where $\alpha \in (n-1, n], n \in \mathbb{N}, n \geq 3, a_{i,}\ \xi_i \in \mathbb{R}, \mathrm{i} = 1, \dots, N\ (N \in \mathbb{N})$, $0 < \xi_1 < \cdots < \xi_N \leq 1,\ p_1, q_1 \in \mathbb{R},\ p_1 \in [1, n-2],\ q_1 \in [0, p_1]$, and $x \in C(0,1) \cap L^1(0,1)$. We denote by $\Delta_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q_1)} \sum_{i=1}^{N} a_i \xi_i^{\alpha-q_1-1}$.

Lemma 1. *If $\Delta_1 \neq 0$, then the function $u \in C[0,1]$ given by*

$$u(t) = \int_0^1 G_1(t,s)x(s)ds, \;\; t \in [0,1], \tag{3}$$

is solution of problem (1)-(2), where

$$G_1(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta_1} \sum_{i=1}^{N} a_i g_2(\xi_i, s), \;\; \forall\, (t,s) \in [0,1] \times [0,1], \tag{4}$$

and

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-p_1-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-p_1-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{5}$$

$$g_2(t,s) = \frac{1}{\Gamma(\alpha-q_1)} \begin{cases} t^{\alpha-q_1-1}(1-s)^{\alpha-p_1-1} - (t-s)^{\alpha-q_1-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-q_1-1}(1-s)^{\alpha-p_1-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 2. *The functions g_1 and g_2 given by (5) have the properties:*

a) $g_1(t,s) \leq h_1(s)$ *for all $t, s \in [0,1]$, where*

$$h_1(s) = \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-p_1-1}(1-(1-s)^{p_1}), \qquad s \in [0,1];$$

b) $g_1(t,s) \geq t^{\alpha-1}h_1(s)$ *for all $t, s \in [0,1]$;*

c) $g_1(t,s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ *for all $t, s \in [0,1]$;*

d) $g_2(t,s) \geq t^{\alpha-q_1-1}h_2(s)$ *for all $t, s \in [0,1]$, where*

$$h_2(s) = \frac{1}{\Gamma(\alpha-q_1)}(1-s)^{\alpha-p_1-1}(1-(1-s)^{p_1-q_1}), \qquad s \in [0,1];$$

e) $g_2(t,s) \leq \frac{1}{\Gamma(\alpha-q_1)} t^{\alpha-q_1-1}$ *for all $t, s \in [0,1]$;*

f) *The functions* g_1 *and* g_2 *are continuous on* $[0,1]\times[0,1]$; $g_1(t,s)\ge 0$, $g_2(t,s)\ge 0$ *for all* $t,s\in[0,1]$; $g_1(t,s)>0$, $g_2(t,s)>0$ *for all* $t,s\in(0,1)$.

Lemma 3. *Assume that* $a_i\ge 0$ *for all* $i=1,\dots,N$ *and* $\Delta_1>0$. *Then the function* G_1 *given by (4) is a nonnegative continuous function on* $[0,1]\times[0,1]$ *and satisfies the inequalities:*

a) $G_1(t,s)\le J_1(s)$ *for all* $t,s\in[0,1]$, *where*
$$J_1(s)=h_1(s)+\frac{1}{\Delta_1}\sum_{i=1}^{N}a_i g_2(\xi_i,s),\ s\in[0,1];$$
b) $G_1(t,s)\ge t^{\alpha-1}J_1(s)$ *for all* $t,s\in[0,1]$;

c) $G_1(t,s)\le \sigma_1 t^{\alpha-1}$, *for all* $t,s\in[0,1]$, *where*
$$\sigma_1=\frac{1}{\Gamma(\alpha)}+\frac{1}{\Delta_1\Gamma(\alpha-q_1)}\sum_{i=1}^{N}a_i\xi_i^{\alpha-q_1-1}.$$

Lemma 4. *Assume that* $a_i\ge 0$ *for all* $i=1,\dots,N$, $\Delta_1>0$, $x\in C(0,1)\cap L^1(0,1)$ *and* $x(t)\ge 0$ *for all* $t\in(0,1)$. *Then the solution* u *of problem (1)-(2) given by (3) satisfies the inequlity* $u(t)\ge t^{\alpha-1}u(t')$ *for all* $t,t'\in[0,1]$.

We can also formulate similar results as Lemmas 1-4 for the fractional boundary value problem
$$D_{0+}^{\beta}v(t)+y(t)=0,\quad 0<t<1, \tag{6}$$
$$v^{(j)}(0)=0,\ j=0,\dots,m-2;\ D_{0+}^{p_2}v(t)|_{t=1}=\sum_{i=1}^{M}b_iD_{0+}^{q_2}v(t)|_{t=\eta_i}, \tag{7}$$
where $\beta\in(m-1,m],m\in\mathbb{N},m\ge 3,b_i,\eta_i\in\mathbb{R},\mathrm{i}=1,\dots,M\ (M\in\mathbb{N})$, $0<\eta_1<\cdots<\eta_M\le 1$, $p_2,q_2\in\mathbb{R}$, $p_2\in[1,m-2]$, $q_2\in[0,p_2]$ and $y\in C(0,1)\cap L^1(0,1)$.

We denote by $\Delta_2,g_3,g_4,G_2,h_3,h_4,J_2$ and σ_2 the corresponding constants and functions for problem (6)-(7) defined in a similar manner as $\Delta_1,g_1,g_2,G_1,\ h_1,h_2,J_1$ and σ_1, respectively. More precisely, we have
$$\Delta_2=\frac{\Gamma(\beta)}{\Gamma(\beta-p_2)}-\frac{\Gamma(\beta)}{\Gamma(\beta-q_2)}\sum_{i=1}^{M}b_i\eta_i^{\beta-q_2-1},$$
$$g_3(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}t^{\beta-1}(1-s)^{\beta-p_2-1}-(t-s)^{\beta-1}, & 0\le s\le t\le 1,\\ t^{\beta-1}(1-s)^{\beta-p_2-1}, & 0\le t\le s\le 1,\end{cases}$$
$$g_4(t,s)=\frac{1}{\Gamma(\beta-q_2)}\begin{cases}t^{\beta-q_2-1}(1-s)^{\beta-p_2-1}-(t-s)^{\beta-q_2-1}, & 0\le s\le t\le 1,\\ t^{\beta-q_2-1}(1-s)^{\beta-p_2-1}, & 0\le t\le s\le 1,\end{cases}$$
$$G_2(t,s)=g_3(t,s)+\frac{t^{\beta-1}}{\Delta_2}\sum_{i=1}^{M}b_ig_4(\eta_i,s),\ \forall\,(t,s)\in[0,1]\times[0,1],$$
$$h_3(s)=\frac{1}{\Gamma(\beta)}(1-s)^{\beta-p_2-1}(1-(1-s)^{p_2}),\qquad s\in[0,1],$$

$$h_4(s) = \frac{1}{\Gamma(\beta - q_2)}(1-s)^{\beta - p_2 - 1}(1-(1-s)^{p_2 - q_2}), \qquad s \in [0,1],$$

$$J_2(s) = h_3(s) + \frac{1}{\Delta_2}\sum_{i=1}^{M} b_i g_4(\eta_i, s), \ \ s \in [0,1],$$
$$\sigma_2 = \frac{1}{\Gamma(\beta)} + \frac{1}{\Delta_2 \Gamma(\beta - q_2)}\sum_{i=1}^{M} b_i {\eta_i}^{\beta - q_2 - 1}.$$

The inequalities from Lemmas 3 and 4 for the functions G_2 and v are the following $G_2(t,s) \leq J_2(s)$, $G_2(t,s) \geq t^{\beta-1}J_2(s)$, $G_2(t,s) \leq \sigma_2 t^{\beta-1}$, for all $t,s \in [0,1]$, and $v(t) \geq t^{\beta-1}v(t^{'})$ for all $t,t^{'} \in [0,1]$.

Remark 1. *Under the assumptions of Lemma 4, and of the corresponding lemma for problem (6)-(7), for $c \in (0,1)$, the solutions u, v of problems (1)-(2) and (6)-(7), respectively, satisfy the inequalities*

$$min_{t\in[c,1]}\, u(t) \geq c^{\alpha-1}\, max_{t^{'}\in[0,1]}\, u(t^{'}),$$
$$min_{t\in[c,1]}\, v(t) \geq c^{\beta-1}\, max_{t^{'}\in[0,1]}\, v(t^{'}).$$

The proofs of our results are based on the following fixed point index theorems. Let E be a real Banach space, $P \subset E$ a cone, "$\leq$" the partial ordering defined by P and θ the zero element in E. For $\varrho > 0$, let B_ϱ=$\{u \in E, \|u\| < \varrho\}$ be the open ball of radius ϱ centered at θ , and its boundary $\partial B_\varrho = \{u \in E, \|u\| = \varrho\}$.

Theorem 1. (Amann, 1976) *Let $A: \bar{B}_\varrho \cap P \to P$ be a completely continuous operator which has no fixed point on $\partial B_\varrho \cap P$. If $\|Au\| \leq \|u\|$ for all $u \in \partial B_\varrho \cap P$, then $i(A, B_\varrho \cap P, P) = 1$.*

Theorem 2. (Amann, 1976) *Let $A: \bar{B}_\varrho \cap P \to P$ be a completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that $u - Au \neq \lambda u_0$, for all $\lambda \geq 0$ and $u \in \partial B_\varrho \cap P$, then $i(A, B_\varrho \cap P, P) = 0$.*

Theorem 3. (Zhou and Xu, 2006) *Let $A: \bar{B}_\varrho \cap P \to P$ be a completely continuous operator which has no fixed point on $\partial B_\varrho \cap P$. If there exists a linear operator $L: P \to P$ and $u_0 \in P \setminus \{\theta\}$ such that*

i) $u_0 \leq Lu_0$, ii) $Lu \leq Au$, $\forall\, u \in \partial B_\varrho \cap P$,

then $i(A, B_\varrho \cap P, P) = 0$.

3. Main Results

In this section we investigate the existence and multiplicity of positive solutions for problem (S)-(BC) under various assumptions on the functions *f* and *g*.

We present the assumptions that we shall use in the sequel.

(H1) $\alpha, \beta \in \mathbb{R},\ \alpha \in (n-1, n],\ \beta \in (m-1, m],\ n, m \in \mathbb{N},\ n, m \geq 3$, $p_1, p_2, q_1, q_2 \in \mathbb{R}, p_1 \in [1, n-2], p_2 \in [1, m-2],\ q_1 \in [0, p_1], q_2 \in [0, p_2]$, $\xi_i \in \mathbb{R},\ a_i \geq 0$ *for all* $i = 1, \dots, N$ $(N \in \mathbb{N})$, $0 < \xi_1 < \cdots < \xi_N \leq 1, \eta_i \in \mathbb{R}$, $b_i \geq 0$ *for all* $i = 1, \dots, M$ $(M \in \mathbb{N}), 0 < \eta_1 < \cdots < \eta_M \leq 1$, $\Delta_1 = \frac{\Gamma(\alpha)}{\Gamma(\alpha - p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha - q_1)} \sum_{i=1}^{N} a_i \xi_i^{\alpha - q_1 - 1} > 0,\ \Delta_2 = \frac{\Gamma(\beta)}{\Gamma(\beta - p_2)} - \frac{\Gamma(\beta)}{\Gamma(\beta - q_2)} \sum_{i=1}^{M} b_i \eta_i^{\beta - q_2 - 1} > 0.$

(H2) *The functions* $f, g: [0,1] \times [0, \infty) \times [0, \infty) \to [0, \infty)$ *are continuous.*

By using Lemma 2, a solution of the following nonlinear system of integral equations

$$\begin{cases} u(t) = \int_0^1 G_1(t,s) f\big(s, u(s), v(s)\big)\, ds, & t \in [0,1], \\ v(t) = \int_0^1 G_2(t,s) g\big(s, u(s), v(s)\big)\, ds, & t \in [0,1] \end{cases}$$

is solution of problem (S)-(BC).

We consider the Banach space $X = C[0,1]$ with supremum norm $\|\cdot\|$ and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cone $P \subset Y$ by $P = \{(u,v) \in Y,\ u(t) \geq 0,\ v(t) \geq 0 \text{ for all } t \in [0,1]\}$.

We introduce the operators $Q_1, Q_2: Y \to X$ and $Q: Y \to Y$ defined by

$$Q_1(u,v)(t) = \int_0^1 G_1(t,s) f\big(s, u(s), v(s)\big)\, ds,\ \ t \in [0,1],$$

$$Q_2(u,v)(t) = \int_0^1 G_2(t,s) g\big(s, u(s), v(s)\big)\, ds,\ \ t \in [0,1],$$

and $Q(u,v) = (\,Q_1(u,v), Q_2(u,v)),\ (u,v) \in Y$.

Under the assumptions (H1) and (H2), it is easy to see that operator $Q: P \to P$ is completely continuous. It is obvious that if *(u,v)* is a fixed point of operator *Q*, then *(u,v)* is a solution of problem (S)-(BC). Therefore, we will study the existence and multiplicity of fixed points of operator *Q*.

Theorem 4. *Assume that (H1) and (H2) hold. If the functions f and g also satisfy the conditions*

(H3) There exist $p \geq 1$ *and* $q \geq 1$ *such that*

$$f_0^s = \lim_{\substack{u+v \to 0 \\ u,v \geq 0}} \sup_{t \in [0,1]} \frac{f(t,u,v)}{(u+v)^p} = 0 \quad \textit{and}$$

$$g_0^s = \lim_{\substack{u+v \to 0 \\ u,v \geq 0}} \sup_{t \in [0,1]} \frac{g(t,u,v)}{(u+v)^q} = 0;$$

(H4) There exists $c \in (0,1)$ *such that*

$$f_\infty^i = \lim_{\substack{u+v \to \infty \\ u,v \geq 0}} \inf_{t \in [c,1]} \frac{f(t,u,v)}{u+v} = \infty \quad \textit{and}$$

$$g_\infty^i = \lim_{\substack{u+v \to \infty \\ u,v \geq 0}} \inf_{t \in [c,1]} \frac{g(t,u,v)}{u+v} = \infty,$$

then problem (S)-(BC) has at least one positive solution $\big(u(t), v(t)\big),\ t \in [0,1]$.

Proof. For c given in (H4) we define the cone

$$P_0 = \left\{(u,v) \in P, \quad \min_{t\in[c,1]} u(t) \geq c^{\alpha-1}\|u\|, \quad \min_{t\in[c,1]} v(t) \geq c^{\beta-1}\|v\|\right\}.$$

From our assumptions and Remark 1, for any $(u,v) \in P$, we deduce that $Q(u,v) = (\,Q_1(u,v), Q_2(u,v)) \in P_0$, that is $Q(P) \subset P_0$.

We consider the functions $u_0, v_0\colon [0,1] \to \mathbb{R}$ defined by

$$u_0(t) = \int_0^1 G_1(t,s)ds, \quad v_0(t) = \int_0^1 G_2(t,s)ds, \quad t \in [0,1],$$

that is (u_0, v_0) is solution of problem (1)-(2) with $x(t) = x_0(t),\ y(t) = y_0(t)$, $x_0(t) = 1,\ y_0(t) = 1$ for all $t \in [0,1]$. Hence $(u_0, v_0) = Q(x_0, y_0) \in P_0$.

We define the set

$$\widetilde{M} = \{(u,v) \in P, \quad \textit{there exists } \lambda \geq 0 \ \textit{ such that } (u,v) = Q(u,v) + \lambda(u_0,v_0)\}.$$

We will show that $\widetilde{M} \subset P_0$ and $\widetilde{M}$ is a bounded set of Y. If $(u,v) \in \widetilde{M}$, then there exists $\lambda \geq 0$ such that $(u,v) = Q(u,v) + \lambda(u_0,v_0)$ or equivalently

$$\begin{cases} u(t) = \int_0^1 G_1(t,s)\big(f\big(s,u(s),v(s)\big) + \lambda\big)\, ds, & t \in [0,1], \\ v(t) = \int_0^1 G_2(t,s)\big(g\big(s,u(s),v(s)\big) + \lambda\big)\, ds, & t \in [0,1]. \end{cases}$$

By Remark 1, we obtain $(u,v) \in P_0$, so $\widetilde{M} \subset P_0$, and

$$\|u\| \leq \frac{1}{c^{\alpha-1}} \min_{t\in[c,1]} u(t), \quad \|v\| \leq \frac{1}{c^{\beta-1}} \min_{t\in[c,1]} v(t), \quad \forall\, (u,v) \in \widetilde{M}. \quad (8)$$

From (H4) we have $f_\infty^i = \infty$ and $g_\infty^i = \infty$. Then for $\epsilon_1 = \frac{2}{c^{\alpha-1}m_1} > 0$, $\epsilon_2 = \frac{2}{c^{\beta-1}m_2} > 0$, there exist $C_1 > 0,\ C_2 > 0$ such that

$$f(t,u,v) \geq \epsilon_1(u+v) - C_1, \quad g(t,u,v) \geq \epsilon_2(u+v) - C_2, \quad (9)$$
$$\forall\, (t,u,v) \in [c,1] \times [0,\infty) \times [0,\infty),$$

where $m_i = \int_c^1 J_i(s)ds$ and $J_i,\ i = 1{,}2$ are defined in Lemma 3.

For $(u,v) \in \widetilde{M}$ and $t \in [c,1]$, by using Lemma 3 and relations (9), we obtain

$$\begin{aligned} u(t) &= Q_1(u,v)(t) + \lambda u_0(t) \geq Q_1(u,v)(t) \\ &= \int_0^1 G_1(t,s) f\big(s,u(s),v(s)\big)ds \geq \int_c^1 t^{\alpha-1} J_1(s) f\big(s,u(s),v(s)\big)ds \\ &\geq c^{\alpha-1}\int_c^1 J_1(s)[\,\epsilon_1\big(u(s)+v(s)\big) - C_1]\, ds \\ &\geq c^{\alpha-1}\epsilon_1 \int_c^1 J_1(s)u(s)ds - c^{\alpha-1}m_1C_1 \\ &\geq c^{\alpha-1}\epsilon_1 m_1 \min_{s\in[c,1]} u(s) - c^{\alpha-1}m_1C_1 \\ &\geq 2\min_{s\in[c,1]} u(s) - C_3, \quad C_3 = c^{\alpha-1}m_1C_1, \end{aligned}$$

and

$$\begin{aligned} v(t) &= Q_2(u,v)(t) + \lambda v_0(t) \geq Q_2(u,v)(t) \\ &= \int_0^1 G_2(t,s) g\big(s,u(s),v(s)\big)ds \geq \int_c^1 t^{\beta-1} J_2(s) g\big(s,u(s),v(s)\big)ds \\ &\geq c^{\beta-1}\int_c^1 J_2(s)[\,\epsilon_2\big(u(s)+v(s)\big) - C_2]\, ds \\ &\geq c^{\beta-1}\epsilon_2 \int_c^1 J_2(s)v(s)ds - c^{\beta-1}m_2C_2 \end{aligned}$$

$\geq c^{\beta-1}\epsilon_2 m_2 \min_{s\in[c,1]} v(s) - c^{\beta-1}m_2 C_2$
$\geq 2\min_{s\in[c,1]} v(s) - C_4,\ \ C_4 = c^{\beta-1}m_2 C_2.$

Therefore, we deduce

$$\min_{t\in[c,1]} u(t) \leq C_3, \quad \min_{t\in[c,1]} v(t) \leq C_4, \ \forall\,(u,v)\in \widetilde{M}. \tag{10}$$

Now from relations (8) and (10), we obtain

$$\|u\| \leq \frac{C_3}{c^{\alpha-1}}, \quad \|v\| \leq \frac{C_4}{c^{\beta-1}},$$

$$\|(u,v)\|_Y = \|u\| + \|v\| \leq \frac{C_3}{c^{\alpha-1}} + \frac{C_4}{c^{\beta-1}} = C_5,$$

for all $(u,v)\in \widetilde{M}$, that is $\widetilde{M}$ is a bounded set of *Y*.

Besides, there exists a sufficiently large $R_1 > 1$ such that

$$(u,v) \neq Q(u,v) + \lambda(u_0,v_0), \ \forall\,(u,v)\in \partial B_{R_1}\cap P, \ \ \forall\ \lambda \geq 0.$$

From (Amann, 1976), we deduce that the fixed point index of operator Q over $B_{R_1}\cap P$ with respect to *P* is

$$i\big(Q, B_{R_1}\cap P, P\big) = 0. \tag{11}$$

Next, from assumption (H3), we conclude that for $\epsilon_3 = \frac{1}{4M_1} > 0$ and $\epsilon_4 = \frac{1}{4M_2} > 0$, there exists $r_1 \in (0,1]$ such that

$$f(t,u,v) \leq \epsilon_3 (u+v)^p, \ \ g(t,u,v) \leq \epsilon_4 (u+v)^q, \tag{12}$$

$$\forall\, t\in[0,1], \ \ u,v\geq 0, \ \ u+v\leq r_1,$$

where $M_i = \int_0^1 J_i(s)ds, \ \ i = 1,2.$

By using (12), we deduce that for all $(u,v)\in \bar{B}_{r_1}\cap P$ and $t\in[0,1]$

$$Q_1(u,v)(t) \leq \int_0^1 J_1(s)\,\epsilon_3\big(u(s)+v(s)\big)^p ds$$
$$\leq \epsilon_3 M_1 \|(u,v)\|_Y{}^p \leq \frac{1}{4}\|(u,v)\|_Y,$$
$$Q_2(u,v)(t) \leq \int_0^1 J_2(s)\,\epsilon_4\big(u(s)+v(s)\big)^q ds$$
$$\leq \epsilon_4 M_2 \|(u,v)\|_Y{}^q \leq \frac{1}{4}\|(u,v)\|_Y.$$

These imply that

$$\|Q_1(u,v)\| \leq \frac{1}{4}\|(u,v)\|_Y, \qquad \|Q_2(u,v)\| \leq \frac{1}{4}\|(u,v)\|_Y,$$

$$\|Q(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| \leq \frac{1}{2}\|(u,v)\|_Y, \ \forall\,(u,v)\in \partial B_{r_1}\cap P.$$

From (Amann, 1976), we conclude that the fixed point index of operator Q over $B_{r_1}\cap P$ with respect to *P* is

$$i\big(Q, B_{r_1}\cap P, P\big) = 1. \tag{13}$$

Combining (11) and (13) we obtain

$$i\big(Q, \big(B_{R_1}\setminus \bar{B}_{r_1}\big)\cap P, P\big) = i\big(Q, B_{R_1}\cap P, P\big) - i\big(Q, B_{r_1}\cap P, P\big) = -1.$$

We deduce that Q has at least one fixed point $(u,v)\in \big(B_{R_1}\setminus \bar{B}_{r_1}\big)\cap P$, that is $r_1 < \|(u,v)\|_Y < R_1$ or $r_1 < \|u\| + \|v\| < R_1$. By Lemma 4, we obtain that $u(t) > 0$ for all $t \in (0,1]$ or $v(t) > 0$ for all $t \in (0,1]$. The proof is completed. ■

Theorem 5. *Assume that (H1) and (H2) hold. If the functions f and g also satisfy the conditions*

(H5) $$f_\infty^s = \lim_{\substack{u+v\to\infty \\ u,v\geq 0}} sup_{\,t\in[0,1]} \frac{f(t,u,v)}{u+v} = 0 \quad \textit{and}$$
$$g_\infty^s = \lim_{\substack{u+v\to\infty \\ u,v\geq 0}} sup_{t\in[0,1]} \frac{g(t,u,v)}{u+v} = 0;$$

(H6) There exist $c \in (0,1)$, $\hat{p} \in (0,1]$ *and* $\hat{q} \in (0,1]$ *such that*
$$f_0^i = \lim_{\substack{u+v\to 0 \\ u,v\geq 0}} inf_{t\in[c,1]} \frac{f(t,u,v)}{(u+v)^{\hat{p}}} = \infty \quad \textit{and}$$
$$g_0^i = \lim_{\substack{u+v\to 0 \\ u,v\geq 0}} inf_{t\in[c,1]} \frac{g(t,u,v)}{(u+v)^{\hat{q}}} = \infty,$$

then problem (S)-(BC) has at least one positive solution $\big(u(t), v(t)\big)$, $t \in [0,1]$.

Proof. From the assumption (H5), we deduce that for $\epsilon_5 = \frac{1}{4M_1} > 0$ and $\epsilon_6 = \frac{1}{4M_2} > 0$, there exist $C_6 > 0$, $C_7 > 0$ such that

$$f(t,u,v) \leq \epsilon_5(u+v) + C_6, \quad g(t,u,v) \leq \epsilon_6(u+v) + C_7, \tag{14}$$
$$\forall\, (t,u,v) \in [0,1]\times[0,\infty)\times[0,\infty).$$

Hence for $(u,v) \in P$, by using (14), we obtain

$$Q_1(u,v)(t) \leq \int_0^1 J_1(s)\big(\epsilon_5\big(u(s)+v(s)\big) + C_6\big)ds$$
$$\leq \epsilon_5(\|u\| + \|v\|)\int_0^1 J_1(s)ds + C_6\int_0^1 J_1(s)ds$$
$$= \epsilon_5\|(u,v)\|_Y M_1 + C_6 M_1 = \frac{1}{4}\|(u,v)\|_Y + C_6 M_1, \quad \forall\, t \in [0,1],$$
$$Q_2(u,v)(t) \leq \int_0^1 J_2(s)\big(\epsilon_6\big(u(s)+v(s)\big) + C_7\big)ds$$
$$\leq \epsilon_6(\|u\| + \|v\|)\int_0^1 J_2(s)ds + C_7\int_0^1 J_2(s)ds$$
$$= \epsilon_6\|(u,v)\|_Y M_2 + C_7 M_2 = \frac{1}{4}\|(u,v)\|_Y + C_7 M_2, \quad \forall\, t \in [0,1],$$

and so

$$\|Q(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| \leq \frac{1}{2}\|(u,v)\|_Y + (C_6 M_1 + C_7 M_2)$$
$$= \frac{1}{2}\|(u,v)\|_Y + C_8, \quad C_8 = C_6 M_1 + C_7 M_2.$$

Then there exists a sufficiently large $R_2 \geq \max\{4C_8, 1\}$ such that

$$\|Q(u,v)\|_Y \leq \frac{3}{4}\|(u,v)\|_Y, \quad \forall\, (u,v) \in P,\ \|(u,v)\|_Y \geq R_2.$$

Hence $\|Q(u,v)\|_Y < \|(u,v)\|_Y$ for all $(u,v) \in \partial B_{R_2} \cap P$ and from (Amann, 1976) we have

$$i\big(Q, B_{R_2} \cap P, P\big) = 1. \tag{15}$$

On the other hand, from (H6) we deduce that for $\epsilon_7 = \frac{1}{c^{\alpha-1}m_1} > 0$, $\epsilon_8 = \frac{1}{c^{\beta-1}m_2} > 0$, there exists $r_2 \in (0,1)$ such that

$$f(t,u,v) \geq \epsilon_7 (u+v)^{\hat{p}}, \quad g(t,u,v) \geq \epsilon_8 (u+v)^{\hat{q}}, \quad (16)$$
$$\forall\, t \in [c,1], \; u,v \geq 0, \; u+v \leq r_2.$$

From (16), we deduce that for any $(u,v) \in \bar{B}_{r_2} \cap P$

$Q_1(u,v)(t) \geq \int_c^1 G_1(t,s) f\big(s,u(s),v(s)\big) ds$
$\geq \int_c^1 \epsilon_7 G_1(t,s)(u(s)+v(s))^{\hat{p}}\, ds$
$\geq \epsilon_7 \int_c^1 G_1(t,s)\big(u(s)+v(s)\big)\, ds =: L_1(u,v)(t), \quad \forall\, t \in [0,1],$
$Q_2(u,v)(t) \geq \int_c^1 G_2(t,s) g\big(s,u(s),v(s)\big) ds$
$\geq \int_c^1 \epsilon_8 G_2(t,s)(u(s)+v(s))^{\hat{q}}\, ds$
$\geq \epsilon_8 \int_c^1 G_2(t,s)\big(u(s)+v(s)\big)\, ds =: L_2(u,v)(t), \quad \forall\, t \in [0,1].$

Hence

$$Q(u,v) \geq L(u,v), \quad \forall\, (u,v) \in \partial B_{r_2} \cap P, \quad (17)$$

where the linear operator $L: P \to P$ is defined by $L(u,v) = \big(L_1(u,v), L_2(u,v)\big)$.

For $(\tilde{u}_0, \tilde{v}_0) \in P \setminus \{(0,0)\}$ defined by

$$\tilde{u}_0(t) = \int_c^1 G_1(t,s) ds, \quad \tilde{v}_0(t) = \int_c^1 G_2(t,s) ds, \quad \forall\, t \in [0,1],$$

we have $L(\tilde{u}_0, \tilde{v}_0) = (L_1(\tilde{u}_0, \tilde{v}_0), L_2(\tilde{u}_0, \tilde{v}_0))$ with

$L_1(\tilde{u}_0, \tilde{v}_0)(t) = \epsilon_7 \int_c^1 G_1(t,s)(\int_c^1 G_1(s,\tau)\, d\tau + \int_c^1 G_2(s,\tau)\, d\tau)\, ds$
$\geq \epsilon_7 \int_c^1 G_1(t,s) \left(\int_c^1 G_1(s,\tau)\, d\tau\right) ds$
$\geq \epsilon_7 \int_c^1 G_1(t,s) \left(\int_c^1 c^{\alpha-1} J_1(\tau)\, d\tau\right) ds$
$= \epsilon_7 c^{\alpha-1} m_1 \int_c^1 G_1(t,s) ds = \int_c^1 G_1(t,s) ds = \tilde{u}_0(t), \; \forall\, t \in [0,1],$
$L_2(\tilde{u}_0, \tilde{v}_0)(t) = \epsilon_8 \int_c^1 G_2(t,s)(\int_c^1 G_1(s,\tau)\, d\tau + \int_c^1 G_2(s,\tau)\, d\tau)\, ds$
$\geq \epsilon_8 \int_c^1 G_2(t,s) \left(\int_c^1 G_2(s,\tau)\, d\tau\right) ds$
$\geq \epsilon_8 \int_c^1 G_2(t,s) \left(\int_c^1 c^{\beta-1} J_2(\tau)\, d\tau\right) ds$
$= \epsilon_8 c^{\beta-1} m_2 \int_c^1 G_2(t,s) ds = \int_c^1 G_2(t,s) ds = \tilde{v}_0(t), \; \forall\, t \in [0,1].$

So

$$L(\tilde{u}_0, \tilde{v}_0) \geq (\tilde{u}_0, \tilde{v}_0). \quad (18)$$

We may suppose that Q has no fixed point on $\partial B_{r_2} \cap P$ (otherwise the proof is finished). From (17), (18) and (Zhou and Xu, 2006, Lemma 3), we conclude that

$$i\big(Q, B_{r_2} \cap P, P\big) = 0. \quad (19)$$

Therefore, from (15) and (19), we have

$$i\big(Q, \big(B_{R_2} \setminus \bar{B}_{r_2}\big) \cap P, P\big) = i\big(Q, B_{R_2} \cap P, P\big) - i\big(Q, B_{r_2} \cap P, P\big) = 1.$$

Then Q has at least one fixed point in $\big(B_{R_2} \setminus \bar{B}_{r_2}\big) \cap P$, that is $r_2 < \|(u,v)\|_Y < R_2$. Thus problem (S)-(BC) has at least one positive solution $(u,v) \in P$. This completes the proof. ■

Theorem 6. *Assume that (H1), (H2), (H4) and (H6) hold. If the functions f and g also satisfy the condition*

(H7) For each $t \in [0,1]$, *f(t,u,v) and g(t,u,v) are nondecreasing with respect to u and v, and there exists a constant* $N_0 > 0$ *such that*

$$f(t, N_0, N_0) < \frac{N_0}{2m_0}, \quad g(t, N_0, N_0) < \frac{N_0}{2m_0}, \quad \forall\, t \in [0,1],$$

where $m_0 = \max\{M_i,\ i = 1,2\}$, $(M_i = \int_0^1 J_i(s)ds,\ i = 1,2)$,

then problem (S)-(BC) has at least two positive solutions $\big(u_1(t), v_1(t)\big)$, $\big(u_2(t), v_2(t)\big)$, $t \in [0,1]$.

Proof. By using (H7), for any $(u, v) \in \partial B_{N_0} \cap P$, we obtain

$$Q_1(u,v)(t) \le \int_0^1 G_1(t,s) f(s, N_0, N_0) ds \le \int_0^1 J_1(s) f(s, N_0, N_0) ds$$
$$< \frac{N_0}{2m_0} \int_0^1 J_1(s) ds = \frac{N_0 M_1}{2m_0} \le \frac{N_0}{2}, \quad \forall t \in [0,1],$$
$$Q_2(u,v)(t) \le \int_0^1 G_2(t,s) g(s, N_0, N_0) ds \le \int_0^1 J_2(s) g(s, N_0, N_0) ds$$
$$< \frac{N_0}{2m_0} \int_0^1 J_2(s) ds = \frac{N_0 M_2}{2m_0} \le \frac{N_0}{2}, \quad \forall t \in [0,1].$$

Then we deduce

$$\|Q(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| < N_0, \quad \forall\, (u,v) \in \partial B_{N_0} \cap P.$$

By (Amann, 1976) we conclude that

$$i\big(Q, B_{N_0} \cap P, P\big) = 1. \tag{20}$$

On the other hand, from (H4) and (H6) and the proofs of Theorem 4 and Theorem 5, we know that there exists a sufficiently large $R_1 > N_0$ and a sufficiently small $r_2 \in (0, N_0)$ such that

$$i\big(Q, B_{R_1} \cap P, P\big) = 0, \quad i\big(Q, B_{r_2} \cap P, P\big) = 0, \tag{21}$$

From the relations (20) and (21), we obtain

$$i\big(Q, \big(B_{R_1} \setminus \bar{B}_{N_0}\big) \cap P, P\big) = i\big(Q, B_{R_1} \cap P, P\big) - i\big(Q, B_{N_0} \cap P, P\big) = -1,$$
$$i\big(Q, \big(B_{N_0} \setminus \bar{B}_{r_2}\big) \cap P, P\big) = i\big(Q, B_{N_0} \cap P, P\big) - i\big(Q, B_{r_2} \cap P, P\big) = 1.$$

Then Q has at least one fixed point $(u_1, v_1) \in \big(B_{R_1} \setminus \bar{B}_{N_0}\big) \cap P$ and has at least one fixed point $(u_2, v_2) \in \big(B_{N_0} \setminus \bar{B}_{r_2}\big) \cap P$. If in Theorem 5, the operator Q has at least one fixed point on $\partial B_{r_2} \cap P$, then by using the first relation from formula above, we deduce that Q has at least one fixed point $(u_1, v_1) \in \big(B_{R_1} \setminus \bar{B}_{N_0}\big) \cap P$ and has at least one fixed point $(u_2, v_2) \in \partial B_{r_2} \cap P$. Therefore, problem (S)-(BC) has two distinct positive solutions (u_1, v_1), (u_2, v_2). The proof is completed. ∎

REFERENCES

Amann H., *Fixed Point Equations and Nonlinear Eigenvalue Problems in Ordered Banach Spaces*, SIAM Review, **18**, 620-709 (1976).

Arafa A.A.M., Rida S.Z., Khalil M., *Fractional Modeling Dynamics of HIV and CD4+ T-Cells During Primary Infection*, Nonlinear Biomed. Phys., **6**, *1*, 1-7 (2012).

Baleanu D., Diethelm K., Scalas E., Trujillo J.J., *Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston, 2012.

Cole K., *Electric Conductance of Biological Systems*, in: Proc. Cold Spring Harbor Symp. Quant. Biol., Col Springer Harbor Laboratory Press, New York, 1993, 107-116.

Das S., *Functional Fractional Calculus for System Identification and Controls*, Springer, New York, 2008.

Ding Y., Ye H., *A Fractional-Order Differential Equation Model of HIV Infection of CD4+ T-Cells*, Math. Comp. Model., **50**, 386-392 (2009).

Djordjevic V., Jaric J., Fabry B., Fredberg J., Stamenovic D., *Fractional Derivatives Embody Essential Features of Cell Rheological Behavior*, Ann. Biomed. Eng., **31,** 692-699 (2003).

Ge Z.M., Ou C.Y., *Chaos Synchronization of Fractional Order Modified Duffing Systems with Parameters Excited by a Chaotic Signal*, Chaos Solitons Fractals, **35,** 705-717 (2008).

Henderson J., Luca R., *Existence and Multiplicity of Positive Solutions for a System of Fractional Boundary Value Problems*, Bound. Value Probl., **60**, 1-17 (2014a).

Henderson J., Luca R., *Positive Solutions for a System of Fractional Differential Equations with Coupled Integral Boundary Conditions*, Appl. Math. Comput., **249**, 182-197 (2014b).

Henderson J., Luca R., *Nonexistence of Positive Solutions for a System of Coupled Fractional Boundary Value Problems*, Bound. Value Probl., **138**, 1-12 (2015).

Henderson J., Luca R., *Positive Solutions for a System of Semipositone Coupled Fractional Boundary Value Problems*, Bound. Value Probl., **61**, 1-23 (2016a).

Henderson J., Luca R., *Boundary Value Problems for Systems of Differential, Difference and Fractional Equations. Positive Solutions*, Elsevier, Amsterdam, 2016b.

Henderson J., Luca R., *Existence of Positive Solutions for a Singular Fractional Boundary Value Problem*, Nonlinear Anal. Model. Control, **22**, *1*, 99-114 (2017a).

Henderson J., Luca R., *Systems of Riemann-Liouville Fractional Equations with Multi-Point Boundary Conditions*, Appl. Math. Comput., **309**, 303-323 (2017b).

Henderson J., Luca R., *Positive Solutions for a System of Fractional Boundary Value Problems*, in Proceedings of EQUADIFF 2017 Conference, Editors Karol Mikula, Daniel Sevcovic, Jozef Urban, Published by Slovak University of Technology, Publishing House, Bratislava, Slovakia, 2017c, 1-10.

Henderson J., Luca R., Tudorache A., *On a System of Fractional Differential Equations with Coupled Integral Boundary Conditions*, Fract. Calc. Appl. Anal., **18**, *2*, 361-386 (2015).

Henderson J., Luca R., Tudorache A., *Positive Solutions for a System of Fractional Differential Equations with Multi-Point Boundary Conditions*, submitted (2017).

Kilbas A.A., Srivastava H.M., Trujillo J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

Klafter J., Lim S.C., Metzler R., *Fractional Dynamics in Physics*, Singapore, World Scientific, 2011.

Luca R., Tudorache A., *Positive Solutions to a System of Semipositone Fractional Boundary Value Problems*, Adv. Difference Equ., **179**, 1-11 (2014).

Metzler R., Klafter J., *The Random Walks Guide to Anomalous Diffusion: a Fractional Dynamics Approach*, Phys. Rep., **339**, 1-77 (2000).

Ostoja-Starzewski M., *Towards Thermoelasticity of Fractal Media*, J. Therm. Stress., **30**, 889-896 (2007).

Podlubny I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

Povstenko Y.Z., *Fractional Thermoelasticity*, New York, Springer, 2015.

Sabatier J., Agrawal O.P., Machado J.A.T., *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

Samko S.G., Kilbas A.A., Marichev O.I., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

Sokolov I.M., Klafter J., Blumen A., *A Fractional Kinetics*, Phys. Today, **55**, 48-54 (2002).

Zhou Y., Xu Y., *Positive Solutions of three-Point Boundary Value Problems for Systems of Nonlinear Second Order Ordinary Differential Equations*, J. Math. Anal. Appl., **320,** 578-590 (2006).

EXISTENŢA SOLUŢIILOR POZITIVE PENTRU O PROBLEMĂ LA LIMITĂ FRACŢIONARĂ

(Rezumat)

Studiem existenţa şi multiplicitatea soluţiilor pozitive pentru sistemul de ecuaţii diferenţiale fracţionare Riemann-Liouville (S) cu neliniarităţi nenegative şi nesingulare, cu condiţiile la limită (BC) cu mai multe puncte care conţin derivate fracţionare.