BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi” din Iaşi
Volumul 64 (68), Numărul 1, 2018
Secţia
MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

ON A "HIDDEN" SYMMETRY OF THE MAXWELL'S EQUATIONS

BY

IRINEL CASIAN BOTEZ[1,*] and MARICEL AGOP[2,3]

"Gheorghe Asachi" Technical University of Iaşi, Romania,
[1]Faculty of Electronics, Telecomunication and Information Technology
[2]Department of Physics
[3]Academy of Romanian Scientists, Bucureşti, Romania

Received: February 26, 2018
Accepted for publication: April 11, 2018

Abstract. It is show that the Maxwell's equations have a "hidden" symmetry on the form of the Barbilian's group. Some properties and implications of this group is also analyzed.

Keywords: Maxwell's equations; Barbilian's group; Jaine's probability.

1. Introduction

Let us consider the Maxwell's equations in simple media (non-dispersive, linear and isotropic) without sources (Harrington, 2001):

$$\begin{aligned}
&\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \\
&\nabla \times \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + \sigma \boldsymbol{E} \\
&\nabla \cdot \boldsymbol{H} = 0 \\
&\nabla \cdot \boldsymbol{E} = 0
\end{aligned} \qquad (1)$$

[*]Corresponding author; *e-mail*: irinel_casian@yahoo.com

Using vectorial calculus, we can transform these equations in two wave equations, one in electric field, $\boldsymbol{E}$, and the other in magnetic field, $\boldsymbol{H}$:

$$\nabla^2 \boldsymbol{E} = \varepsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu\sigma \frac{\partial \boldsymbol{E}}{\partial t} \tag{2}$$

$$\nabla^2 \boldsymbol{H} = \varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} + \mu\sigma \frac{\partial \boldsymbol{H}}{\partial t} \tag{3}$$

We only continue with the equation in electric field, since the equation in magnetic field has the same form.

In Cartesian coordinate systems, the vectorial Eq. (2) is equivalent with 3 similar scalar equations:

$$\nabla^2 E_i = \varepsilon\mu \frac{\partial^2 E_i}{\partial t^2} + \mu\sigma \frac{\partial E_i}{\partial t}, \quad i = x, y, z \tag{4}$$

For this equation a “hidden” symmetry in the form of Barbilian’s group is given.

2. Mathematical Model

Every component is a scalar function of space and time. Following the method of variables separation, we consider:

$$E_i(x,y,z,t) = g_i(x,y,z) T(t), \quad i = x, y, z \tag{5}$$

So, the Eq. (4) become:

$$\nabla^2 g_i - \lambda_i g_i = 0 \tag{6}$$

$$\frac{d^2 T}{dt^2} + \frac{\sigma}{\varepsilon}\frac{dT}{dt} - \frac{\lambda_i}{\mu} T = 0 \tag{7}$$

Now, we restrain the problem to one-dimensional (1D) case, *i.e.* that the electric field has component only in x-direction. In this situation, $g_i(x,y,z) = \theta(x)$ and the Eqs. (6) and (7) become:

$$\frac{d^2\theta}{dx^2} + k_0^2 \theta(x) = 0 \tag{8}$$

$$\frac{d^2T}{dt^2}+\frac{\sigma}{\varepsilon}\frac{dT}{dt}+\frac{k_0^2}{\mu}T=0 \tag{9}$$

where

$$-\lambda_i=k_0^2$$

The most general solution of the Eq. (8) can be written in the form:

$$\theta(x)=he^{i(k_0x+\varphi)}+\overline{h}e^{-i(k_0x+\varphi)} \tag{10}$$

with h a complex amplitude, $\overline{h}$ its complex conjugate and φ a phase.

This solution describes a complex system structural units (electrical field – material structures) of the same "characteristic" k_0, in which the structural unit is identified by means of the parameters $h,\overline{h}$ and $k=e^{i\varphi}$. Now, a question arises. Which is the relation among the structural units of the complex system having the same k_0? The mathematical answer to this question can be obtained if we admit that all we intend here is to find a way to switch from a triplet of numbers - the initial conditions - of a structural unit, to the same triplet of another structural unit having the same k_0.

This passage implies a "hidden symmetry" which is made explicit in the form of a continuous group with three parameters, group that is simple transitive and which can be constructed using a certain definition of k_0.

We start from the idea that the ratio between two fundamental solutions of Eq. (8) is a solution of Schwartz's nonlinear equation (Mihăileanu, 1972):

$$\{\tau_0(x),x\}=2k_0^2 \quad , \quad \tau_0(x)=e^{-2ik_0x} \tag{11}$$

where the curly brackets define Schwartz's derivative of τ_0 with respect to *x*,

$$\{\tau_0(x),x\}=\partial_x\left(\frac{\partial_{xx}\tau_0}{\partial_x\tau_0}\right)-\frac{1}{2}\left(\frac{\partial_{xx}\tau_0}{\partial_x\tau_0}\right)^2 \tag{12}$$

This equation proves to be a veritable definition of k_0, as a general characteristic of a complex system of structural units which can be swept through a continuous group with three parameters - the homographic group.

Indeed, Eq. (11) is invariant with respect to the dependent variable change:

$$\tau(x) = \frac{a\tau_0(x) + b}{c\tau_0(x) + d}, \quad a, b, c, d \in \mathbb{R} \tag{13}$$

and this statement can be directly verified.

In this way, $\tau(x)$ characterizes another structural unit of the same k_0, which allows us to state that, starting from a standard structural unit, we can sweep the entire complex system of structural units having the same k_0, when we are not conditioning (we leave it free) the three ratios $a : b : c : d$ in Eq. (13).

We can make even more accurate the correspondence between a homographic transformation and a structural unit of the complex system, by associating to every structural unit of the complex system, a "personal" $\tau(x)$ by the relation:

$$\tau_1(x) = \frac{h + \bar{h}k\tau_0(x)}{1 + k\tau_0(x)}, \quad k = e^{-2i\phi} \tag{14}$$

Let us observe that τ_0 and τ_1 can be used freely one in place of another and this leads us to the following transformation group for the initial conditions:

$$h \leftrightarrow \frac{ah + b}{ch + d}, \bar{h} \leftrightarrow \frac{a\bar{h} + b}{c\bar{h} + d}, k \leftrightarrow \frac{c\bar{h} + d}{ch + d}k \tag{15}$$

This group is simple transitive: to a given set of values $(a/c, b/c, d/c)$ will correspond a single transformation and only one of the group.

The group (15) works as a group of "synchronization" among the various structural units of the complex system, process to which the amplitudes and phases of each of them obviously participate, in the sense that they ate correlated, too. More precisely, by means of (15), the phase of k is only moved with a quantity depending on the amplitude of the structural unit of complex system at the transition among various structural units of the complex system. But not only that, the amplitude of the structural unit of the complex system is also affected homographically.

The usual "synchronization" manifested through the delay of the amplitudes and phases of the structural units of the complex system must represent here only a totally particular case.

Theorem 1: *In the "field variables" space of the synchronization group one can a priori build a probabilistic theory based on its elementary measure, as an elementary probability. Then the invariant function of the synchronization group becomes the repartition density of an elementary probability.*

The proof of these statements is based on the differential and integral properties of the homographic group. Thus, considering a specific parametrization of the group (15), the infinitesimal generators (Mercheș and Agop, 2015):

$$\hat{B}_1 = \frac{\partial}{\partial h} + \frac{\partial}{\partial \bar{h}}, \hat{B}_2 = h\frac{\partial}{\partial h} + \bar{h}\frac{\partial}{\partial \bar{h}}, \hat{B}_3 = h^2\frac{\partial}{\partial h} + \bar{h}^2\frac{\partial}{\partial \bar{h}} + \left(h - \bar{h}\right)k\frac{\partial}{\partial k} \tag{16}$$

satisfy the commutation relations:

$$\left[\hat{B}_1, \hat{B}_2\right] = \hat{B}_1, \left[\hat{B}_2, \hat{B}_3\right] = \hat{B}_3, \left[\hat{B}_3, \hat{B}_1\right] = -2\hat{B}_2 \tag{17}$$

The structure of the group (15) is given by Eq. (17) so that the only non-zero structure constants should be:

$$C_{12}^1 = C_{23}^3 = -1, C_{31}^2 = -2 \tag{18}$$

Therefore, the invariant quadratic from is given by the “quadratic” tensor of the group (15):

$$C_{\alpha\beta} = C_{\alpha\nu}^{\mu} C_{\beta\mu}^{\nu} \tag{19}$$

where summation over repeated indices is understood. Using (18) and (19), the tensor $C_{\alpha\beta}$ writes:

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 0 \end{pmatrix} \tag{20}$$

meaning that the invariant metric of the group (15) has the form:

$$\frac{ds^2}{g^2} = \omega_0^2 - 4\omega_1\omega_2 \tag{21}$$

with g an arbitrary factor and ω_α , $\alpha = 1,2,3$ three differential 1-forms (Flanders, 1989), absolutely invariant through the group (15). Barbilian takes these 1-forms as being given by the relations (Barbilian, 1937; Mercheș and Agop, 2015):

$$\omega_0 = -i\left(\frac{dk}{k} - \frac{dh + d\bar{h}}{h - \bar{h}}\right), \omega_1 = \frac{dh}{\left(h - \bar{h}\right)k}, \omega_2 = \frac{-kd\bar{h}}{\left(h - \bar{h}\right)} \quad (22)$$

so that the metric (21) becomes

$$\frac{ds^2}{g^2} = -\left(\frac{dk}{k} - \frac{dh + d\bar{h}}{h - \bar{h}}\right)^2 + 4\frac{dhd\bar{h}}{\left(h - \bar{h}\right)^2} \quad (23)$$

It is worthwhile to mention a property connected to the integral geometry: the group (15) is measurable. Indeed, it is simply transitive and, since its structure vector:

$$C_\alpha = C^\nu_{\nu\alpha} \quad (24)$$

is identically null, as it can be seen from (18) , this means that it possess the invariant function:

$$F\left(h, \bar{h}, k\right) = -\frac{1}{\left(h - \bar{h}\right)^2 k} \quad (25)$$

which is the inverse of the modulus of determinant of a linear system obtained on the basis of infinitesimal transformations of the group (15).

As a result, in the space of the field variables $\left(h, \bar{h}, k\right)$ one can *a priori* construct a probabilistic theory in the sense of Jaynes (on the circumstances left unspecified in an experiment), based on the elementary measure of the group (15):

$$dP\left(h, \bar{h}, k\right) = -\frac{dh \wedge d\bar{h} \wedge dk}{\left(h - \bar{h}\right)^2 k} \quad (26)$$

as elementary probability, where $\wedge$ denotes the external product of the 1-forms. In such context, the invariant function of the group (15), *i.e.* relation (25), becomes the repartition density of the elementary probability (26). An attitude toward Quantum Mechanics which is suitable for Quantum Gravity in general, and for its application to cosmology in particular, is not so easy to find. A philosophically realistic attitude toward Quantum Mechanics would seem to be more effective than one based on operators which must find their physical meaning in terms of measurements. Where Quantum Theory differs from Classical Mechanics (in this view) is in its dynamics, which of course is stochastic rather than deterministic. As such, the theory functions by furnishing probabilities for sets of histories. What ordinarily makes it difficult to regard Quantum Mechanics as in essence a modified form of probability theory, is the peculiar fact that it works with complex amplitudes rather than directly with probabilities, the former being more like square roots of the latter. In this context the above mentioned whole arsenal of Quantum Mechanics can be extended to fractal manifolds by means of a Jaynes type procedure (Jaynes, 1973).

The above results can be re-written in real terms based on the transformation:

$$\left(h,\bar{h},k\right)\to\left(u,v,\phi\right) \tag{27}$$

which can be made explicit through the relations

$$h=u+iv,\bar{h}=u-iv,k=e^{i\phi} \tag{28}$$

Thus, both the operators (16) and the 1-forms (22) have the expressions:

$$\hat{M}_1=\frac{\partial}{\partial u},\hat{M}_2=u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v},\hat{M}_3=\left(u^2-v^2\right)\frac{\partial}{\partial u}+2uv\frac{\partial}{\partial v}+2v\frac{\partial}{\partial\phi} \tag{29}$$

Respectively

$$\begin{gathered}\Omega^1\equiv\omega^0=d\phi+\frac{du}{v},\Omega^2=\omega^1=cos\,\phi\frac{du}{v}+sin\,\phi\frac{dv}{v},\Omega^3=\\=\omega^2=-sin\,\phi\frac{du}{v}+cos\,\phi\frac{dv}{v}\end{gathered} \tag{30}$$

while the 2-form (23) reduces to the two-dimensional Lorentz metric

$$-\left(\Omega^1\right)^2+\left(\Omega^2\right)^2+\left(\Omega^3\right)^2=-\left(d\phi+\frac{du}{v}\right)^2+\frac{du^2+dv^2}{v^2} \tag{31}$$

Theorem 2: *The existence of a transport of directions in the Levi-Civita sense in the field variables space substitutes the homographic group with that of spin as a synchronization group*.

Let us focus on the metric (23) or (31). It is reduced to the metric of Lobachewski's plane in Poincare's representation:

$$\frac{ds^2}{g^2}=4\frac{dhd\bar{h}}{\left(h-\bar{h}\right)^2} \tag{32}$$

for the condition $\omega_0=0$, *i.e.*, in real terms (28)

$$d\phi=-\frac{du}{v} \tag{33}$$

Since by this restriction the metric (31) in the variables (28) reduces to Lobachewski's one in Beltrami's representation:

$$\frac{ds^2}{g^2}=-\frac{du^2+dv^2}{v^2} \tag{34}$$

the condition (33) defines a parallel transport of vectors in the sense of Levi-Civita (the definition of the parallelism angle in the Lobachewski plane, that is, the form of connection (Agop *et al*., 2015; Mercheș and Agop, 2015): the application point of the vector moves on the geodesic, the vector always making a constant angle with the tangent to the geodesic in the current point. Indeed, taking advantage of the fact that the metric of the plane is conformal Euclidean, we can calculate the angle between the initial vector and the vector transported through parallelism, as the integral of the equation (Agop *et al*., 2015; Mercheș and Agop, 2015).

$$d\phi=\frac{1}{2}\left[\frac{\partial}{\partial v}\left(ln\,F\right)du-\frac{\partial}{\partial u}\left(ln\,F\right)dv\right],F\left(u,v\right)=\frac{1}{v^2} \tag{35}$$

along the transport curve.

Since F (u, v) represents the conformal factor of the given metric, introducing it in (35), we find (33).

The "ensemble" of the initial conditions of the structural units of the complex system corresponding to the same k_0 can be organized as a geometry of the hyperbolic plane. More precisely, these structural units of the complex system correspond to a situation where their initial conditions can be chosen from among points of a hyperbolic plane.

The existence of the parallel transport in the sense of Levi-Civita (33) implies either the substitution of the operators (16) with the operators:

$$\hat{B}_1' = \frac{\partial}{\partial h} + \frac{\partial}{\partial \bar{h}}, \hat{B}_2' = h\frac{\partial}{\partial h} + \bar{h}\frac{\partial}{\partial \bar{h}}, \hat{B}_3' = h^2\frac{\partial}{\partial h} + \bar{h}^2\frac{\partial}{\partial \bar{h}} \tag{36}$$

in the case of the representation in complex variables, or the substitution of the operators (29) with the operators:

$$\hat{M}_1' = \frac{\partial}{\partial u}, \hat{M}_2' = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \hat{M}_3' = \left(u^2 - v^2\right)\frac{\partial}{\partial u} + 2uv\frac{\partial}{\partial v} \tag{37}$$

in the case of the representation in real variables.

Theorem 3: *Through the correlation phase-amplitude given by the relation* (33), *the operators* (37) *reduce to the spin operators in the null vectors space*

$$\hat{S}_1 = cos\,\psi v\frac{\partial}{\partial v} - sin\,\psi\frac{\partial}{\partial \psi}, \hat{S}_2 = sin\,\psi v\frac{\partial}{\partial v} + cos\,\psi\frac{\partial}{\partial \psi}, \hat{S}_3 = i\frac{\partial}{\partial \psi} \tag{38}$$

Precisely, we discuss about the compactifcation of the angular momentum in the null vectors space in the form of the spin.

These operators multiplied with the factor $\lambda\left(dt\right)^{(2/D_F)-1}$, are identical, with the fractal angular momentum operators in the representations:

$$x = v\,sin\,\psi, y = -v\,cos\,\psi, z = iv \tag{39}$$

One can directly verify that, abstraction by a constant factor, the operators (38) are just the fractal spin operators satisfying the same commutation relations as Pauli matrix $\sigma_i\left(i = 1,2,3\right)$. They can be interpreted as fractal angular momentum operators in the fractal space of null radius

$$x^2 + y^2 + z^2 = 0 \tag{40}$$

The corresponding variables (v, ψ) are not concrete variables but just only internal freedom degrees. Moreover, the differential and integral geometry of this group imply the “explanation of the circumstances left unspecified in an experiment” in the Jaynes probabilistic theory, while the compactification of the angular momentum in the null vectors space through the definition of a parallel transport on directions in the Levi-Civita sense in a hyperbolic space implies the spin.

3. Conclusions

It is shown that the Maxwell's equations have a "hidden" symmetry in the form of Barbilian's group. In such conjecture, some implications and properties of this group are given.

REFERENCES

Agop M., Gavriluț A. *et al*., *Implications of Onicescu's Informational Energy in Some Fundamental Physical Models*, International Journal of Modern Physics B 29(1550045) (2015).

Barbilian D., *Die von einer Quantik induzierte Riemannsche Metrik (in German)*, Comptes Rendus de l'Academie Roumaine des Sciences, **2**, 198 (1937).

Flanders H., *Differential Forms with Applications to the Physical Science*, New York, Dover Publication (1989).

Harrington R.F., Time-Harmonic Electromagnetic Fields, New York, IEEE Press (2001).

Jaynes E.T., *The Well Posed Problem*, Foundations of Physics, **3**, 477-493 (1973).

Mercheș I., Agop M., *Differentiability and Fractality in Dynamics of Physical Systems*, World Scientific (2015).

Mihăileanu M., *Differential, Projective and Analytical Geometry (in Romanian)*. Bucharest, Didactic and Pedagogical Publishing House (1972).

ASUPRA UNEI SIMETRII „ASCUNSE" A ECUAŢIILOR LUI MAXWELL

(Rezumat)

Se arată că ecuaţiile cîmpului electromagnetic prezintă o simetrie „ascunsă" ce se poate explicita sub forma grupului de invariantă Barbilian. Într-o asemenea conjunctură, câteva proprietăţi şi implicaţii ale acestui grup sunt de asemenea date.