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VARIANTS OF ATOMICITY AND SOME PHYSICAL APPLICATIONS

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Abstract. In this paper, some results concerning various forms of atomicity are given from the Quantum Measure Theory mathematical perspective and several physical applications are provided. Precisely, the mathematical concept of minimal atomicity is extended, and, based on the remark that Quantum Mechanics is a particular case of Fractal Mechanics for a specified scale resolution, the concept of fractal atomicity (and, particularly, fractal minimal atomicity) is introduced. Some of their mathematical properties are also given.

Keywords: Atom; Pseudo-atom; Minimal atom; Fractal atom; Null-additive set (multi)function.

1. Introduction

Measure Theory concerns with assigning a notion of size to sets. In the last years, non-additive measures theory was given an increasing interest due to its various applications in a wide range of areas. It is used to describe situations concerning conflicts or cooperations among intelligent rational players, giving an appropriate mathematical framework to predict the outcome of the process. Precisely, theories dealing with (pseudo)atoms and monotonicity are used in

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statistics, game theory, probabilities, artificial intelligence. The notion of non-atomicity for set (multi)functions plays a key role in measure theory and its applications and extensions. Even just replacing-additivity with finite additivity for measures requires some stronger non-atomicity property for the same conclusion to hold.

(Non)atomic measures and purely atomic measures have been investigated (in different variants) due to their special form and their special properties, *e.g.* (Chiţescu, 1975, 2001; Cavaliere and Ventriglia, 2014; Gavriluţ and Agop, 2016; Gavriluţ and Croitoru, 2008, 2009, 2010; Gavriluţ, 2010, 2011, 2012; Gavriluţ *et al.*, 2015; Khare and Singh, 2008; Li *et al.*, 2014, 2015; Pap, 1994, 1995, 2002; Pap *et al.* 2016; Rao and Rao, 1983; Suzuki, 1991; Wu and Bo, 2007).

One important application of Measure Theory is in probability, where a measurable set is interpreted as an event and its measure as the probability that the event will occur. Since probability is an important notion in Quantum Mechanics, Measure Theory's techniques could be used to study quantum phenomena. Unfortunately, one of the foundational axioms of Measure Theory does not remain valid in its intuitive application to Quantum Mechanics.

Although classical measure theory imposes strict additivity conditions, a rich theory of non-additive measures developed. Precisely, modifications of traditional Measure Theory (Pap, 1994, 1995, 2002) led to Quantum Measure Theory (Gudder, 2009a, 2009b, 2010, 2011a, 2011b; Salgado, 2002; Sorkin, 1994, 1997, 2007; Surya and Waldlden, 2008). Practically, an extended notion of a measure has been introduced and its applications to the study of interference, probability, and space-time histories in Quantum Mechanics have been discussed (Schweizer and Sklar, 1983).

Quantum Measure Theory is a generalization of Quantum Theory where physical predictions are computed from a matrix known as a decoherence functional. Introduced by (Sorkin, 1994, 1997, 2007), quantum measures help us to describe Quantum Mechanics and its applications to Quantum Gravity and Cosmology (Hartle, 1990). Quantum Measure Theory indicates a wide variety of applications, its mathematical structure being used in the standard quantum formalism.

Despite the continuous efforts of numerous scientists, reconciling General Relativity with Quantum Theory remains one of the most important open problems in Physics. The framework of General Relativity suggests that one promising approach to such unification will be by means of a reformulation of Quantum Theory in terms of histories rather than states. Following this idea, (Sorkin, 1994, 1997, 2007), has proposed a history-based framework, which can unify standard Quantum Mechanics as well as physical theories beyond the quantum formalism.

In such framework, Schrödinger's equation from Quantum Mechanics can be identified with a particular type of geodesic of the fractal space. In consequence, fundamental concepts of Quantum Mechanics can be extended to similar concepts, but on fractal manifolds. The aim of this paper is to provide the mathematical-physical framework that is necessary to extend some of these concepts. Precisely, we extend the concept of atoms/pseudo-atoms to the concept of fractal minimal atom/fractal pseudo-atom, respectively. We also give characterizations from a mathematical viewpoint to these new concepts and we make explicit certain physical implications. The notion of a fractal minimal atom as a particular case of fractal atom is also discussed.

2. Towards Quantum Measure Theory by Means of Fractal Mechanics

The basic idea behind Quantum Measure Theory, or Generalized Quantum Mechanics, for that matter, is to provide a description of the world in terms of histories. A history is a classical description of the system under consideration for a given period of time, finite or infinite. If we are trying to describe a system of N particles, then a history will be given by N classical trajectories. If we are working with a field theory, then a history will correspond to the spatial configuration of the field as a function of time. In either case, Quantum Measure Theory tries to provide a way to describe the world through classical histories by extending the notion of probability theory which is clearly not rich enough to model our universe.

On the other hand, structures, self-structures etc. of the Nature can be assimilated to complex systems, taking into account both their functionality, as well as their structure (Mitchell, 2009; Nottale, 2011). The models commonly used to study the dynamics of complex systems are based on the assumption, otherwise unjustified, of the differentiability of the physical quantities that describe it, such as density, momentum, energy etc. (for mathematical models and for applications, see (Mercheş and Agop, 2015; Nottale, 2011).

The success of differentiable models must be understood sequentially, *i.e.* on domains large enough that differentiability and integrability are valid. But differential method fails when facing the physical reality, with non-differentiable or non-integral physical dynamics, such as instabilities in the case of dynamics of complex systems, instabilities that can generate both chaos and patterns.

In order to describe such dynamics of complex systems, but still remaining tributary to a differential hypothesis, it is necessary to introduce, in an explicit manner, the scale resolution in the expressions of the physical variables that describe these dynamics and, implicitly, in the fundamental equations of "evolution" (for example, density, momentum, energy equations etc.). This means that any dynamic variable, dependent, in a classical meaning, on both spatial coordinates and time (Michel and Thomas, 2012; Mitchell, 2009), becomes, in this new context, dependent also on the resolution scale.

In other words, instead of working with a dynamic variable, described through a strictly non-differentiable mathematical function, we will just work with different approximations of that function, derived through its averaging at different resolution scales. Consequently, any dynamic variable acts as the limit of a functions family, the function being non-differentiable for a null resolution scale and differentiable for a non-zero resolution scale.

This approach, well adapted for applications in the field of dynamics of complex systems, where any real determination is conducted at a finite resolution scale, clearly implies the development both of a new geometric structure and of a physical theory (applied to dynamics of complex systems) for which the motion laws, invariant to spatial and temporal coordinates transformations, are integrated with scale laws, invariant at scale transformations.

Such a theory that includes the geometric structure based on the above presented assumptions was developed in the Scale Relativity Theory (Nottale, 2011) and more recently in the Scale Relativity Theory with an arbitrary constant fractal dimension (Mercheş and Agop, 2015). Both theories define the "fractal physics models" class (Mercheş and Agop, 2015; Nottale, 2011).

Various theoretical aspects and applications of the Scale Relativity Theory with an arbitrary constant fractal dimension in the field of physics are presented in (Mercheş and Agop, 2015; Nottale, 2011). In this model, if we assume that the complexity of interactions in the dynamics of complex systems is replaced by non-differentiability, then the motions constrained on continuous, but differentiable curves in an Euclidean space are replaced with free motions, without any constrains, on continuous, but non-differentiable curves (fractal curves) in a fractal space. In other words, for time resolution scale that prove to be large when compared with the inverse of the highest Lyapunov exponent (Mandelbrot, 1983), the deterministic trajectories are replaced by a collection of potential routes, so that the concept of "definite positions" is substituted by that of an ensemble of positions having a definite probability density (Mandelbrot, 1983; Mercheş and Agop, 2015; Nottale, 2011).

In consequence, the motion curves have double identity: both geodesics of the fractal space and streamlines of a fractal fluid, whose entities (the structural units of the complex system) are substituted with the geodesics themselves so that any external constrains are interpreted as a selection of geodesics by means of measuring device.

Since in such conjecture the Quantum Mechanics becomes a particular case of Fractal Mechanics, then Quantum Measure Theory could become, in our opinion, a particular type of a Fractal Measure Theory.

3. Minimal Atoms

Let T be an abstract nonvoid set, $\mathcal{C}$ a ring of subsets of T, X a Banach space, $\mathcal{P}_f(X)$ the family of all nonvoid closed subsets of X and $\mu: \mathcal{C} \to \mathcal{P}_f(X)$ an arbitrary set multifunction which satisfies the condition $\mu(\emptyset) = \{0\}$.

By $|\mu|$, defined on $\mathcal{C}$ and taking values in $[0,\infty]$, we mean the set function defined for every $A \in \mathcal{C}$ by $|\mu(A)| = h(\mu(A),\{0\})$, where h is the Hausdorff-Pompeiu pseudo-metric (Gavriluţ, 2012).

Definition 3.1 I) We say that μ is:

(i) *monotone* with respect to the inclusion of sets if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$;

(ii) *null-additive* if $\mu(A \cup B) = \mu(A)$, for every $A, B \in \mathcal{C}$, with $\mu(B) = 0$;

(iii) *null-null-additive* if $\mu(A \cup B) = \{0\}$, for every $A, B \in \mathcal{C}$, with $\mu(A) = \mu(B) = \{0\}$.

II) We say that a set $A \in \mathcal{C}$ is:

(i) *a minimal atom of* μ if $\mu(A) \supseteq \{0\}$, $\mu(A) \neq \{0\}$ and for every $B \in \mathcal{C}, B \subseteq A$, we have either $\mu(B) = \{0\}$ or $A = B$;

(ii) (Gavriluţ, 2010, 2011, 2012; Gavriluţ and Croitoru, 2008, 2009, 2010) *an atom of* μ if $\mu(A) \supseteq \{0\}$, $\mu(A) \neq \{0\}$ and for every $B \in \mathcal{C}, B \subseteq A$, we have either $\mu(B) = \{0\}$ or $\mu(A \setminus B) = \{0\}$;

(iii) (Gavriluţ, 2010, 2011, 2012; Gavriluţ and Croitoru, 2008, 2009, 2010) *a pseudo-atom of* μ if $\mu(A) \supseteq \{0\}$, $\mu(A) \neq \{0\}$ and for every $B \in \mathcal{C}, B \subseteq A$, we have either $\mu(B) = \{0\}$ or $\mu(A) = \mu(B)$.

Obviously, there exist atoms which are not minimal atoms.

We denote by $\mathcal{A}$ the collection of all atoms of μ and by $\mathcal{MA}$ the collection of all minimal atoms of μ. In what follows, suppose that μ is *monotone*.

Remark 3.2 (i) Any minimal atom is also an atom (and a pseudo-atom), so,

$\mathcal{MA} = \{A \in \mathcal{C}; \mu(A) \supseteq \{0\}, \quad \mu(A) \neq \{0\}$ and for every $B \in \mathcal{C}, B \subseteq A, B \neq A$ we have $\mu(B) = \{0\}\} \subseteq$

$\subseteq \mathcal{A} = \{A \in \mathcal{C}; \mu(A) \supseteq \{0\}, \mu(A) \neq \{0\}$ and for every $B \in \mathcal{C}, B \subseteq A$ we have either $\mu(B) = \{0\}$ or $\mu(A \setminus B) = \{0\}\}$;

(ii) If, moreover, μ is null-additive, then any atom of μ is also a pseudo-atom;

(iii) If A is a minimal atom of μ, then for every $B \in \mathcal{C}, B \subseteq A, B \neq A$ we have $\mu(B) \supseteq \{0\}, \mu(B) \neq \{0\}$;

(iv) If $m : \mathcal{C} \to [0, \infty)$ is monotone, $m(\emptyset) = 0$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{C}$, then a set $A \in \mathcal{C}$ is an atom / pseudo-atom / minimal atom of μ if and only if the same is A for m in the sense of (Mesiar *et al*., 2017; Ouyang *et al*., 2015).

μ is called *the set multifunction induced by the set function* m.

In consequence, one can have different examples concerning minimal atoms with respect to the set multifunction induced by a set function, taking as starting point the examples given in (Mesiar *et al*., 2017; Ouyang *et al*., 2015).

Proposition 3.3 *If* $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ *is null-null-additive and* $A, B \in \mathcal{C}$ *are two different minimal atoms of* μ*, then* $A \cap B = \emptyset$.

Proof. Suppose that, on the contrary, there exist two non-disjoint, different minimal atoms $A, B \in \mathcal{C}$ of μ. Since $A \setminus (A \cap B) = A \setminus B \subseteq A$ and $A \cap B \subseteq B$, then $[\mu(A \setminus B) = \{0\}$ or $A \setminus B = A]$ and $[\mu(A \cap B) = \{0\}$ or $A \cap B = B]$.

(i) If $\mu(A \setminus B) = \{0\}, \mu(A \cap B) = \{0\}$, since μ is null-null-additive, we get that $\mu(A) = \{0\}$, a contradiction.

(ii) If $A \setminus B = A$, then $A \cap B = \emptyset$, a contradiction.

(iii) If $\mu(A \setminus B) = \{0\}$ and $A \cap B = B$, then $B \subseteq A$, so $\mu(B) = \{0\}$ (or $B = A$, which is false), so again by the null-null-additivity of μ, we have $\mu(A) = \{0\}$, a contradiction.

Evidently, if $A \in \mathcal{C}$ is a minimal atom of μ, it can not exist another different minimal atom $A_1 \in \mathcal{C}$ of μ so that $A_1 \subset A$.

Proposition 3.4 *(i) If* T *is finite, then for every* $A \in \mathcal{C}$, *with* $\mu(A) \supseteq \{0\}, \mu(A) \neq \{0\}$, *there exists* $B \in \mathcal{C}, B \subseteq A$, *which is a minimal atom of* μ.

(ii) If, moreover, A *is an atom of* μ *and* μ *is null-additive, then* $\mu(A) = \mu(B)$ *and the set* B *is unique.*

Proof. (i) Let us consider the collection of sets $\mathcal{M} = \{M \in \mathcal{C}, M \subseteq A, \mu(M) \supseteq \{0\}, \mu(M) \neq \{0\}\}$. Obviously, $\mathcal{M} \neq \emptyset$, since $A \in \mathcal{C}$. We remark that any minimal element of $\mathcal{M}$ is a minimal atom of μ. Indeed, let $M \in \mathcal{M}$ be a minimal element of $\mathcal{M}$. Evidently, there can not exist $D \in \mathcal{M}$ so that $D \subseteq M$ and $D \neq M$ (*).

Since $M \in \mathcal{M}$, then $M \in \mathcal{C}, M \subseteq A, \mu(M) \supseteq \{0\}, \mu(M) \neq \{0\}$.

We demonstrate that M is a minimal atom of μ . Indeed, for any $S \subseteq M$, $S \in \mathcal{C}$, we have either $\mu(S) = \{0\}$ or $\mu(S) \supseteq \{0\}, \mu(S) \neq \{0\}$. In the latter case, we have either $S = M$ or $S \neq M$, which is in contradiction with $(*)$.

(ii) If on the contrary there are two different minimal atoms B_1 and B_2 of μ , then $\mu(A \setminus B_1) = \mu(A \setminus B_2) = \{0\}$, whence $\mu(A) = \{0\}$, a contradiction.

Proposition 3.5 (self-similarity of minimal atoms). *Any subset* $B \in \mathcal{C}$, *with* $\mu(B) \supseteq \{0\}, \mu(B) \neq \{0\}$ *of a minimal atom* $A \in \mathcal{C}$ *of* μ *is a minimal atom of* μ, *too.*

Proof. Let $A \in \mathcal{C}$ be a minimal atom of μ and consider any $B \in \mathcal{C}$, with $\mu(B) \supseteq \{0\}, \mu(B) \neq \{0\}$, $B \subseteq A$. We prove that B is a minimal atom of μ. Indeed, for any $C \in \mathcal{C}, C \subseteq B$, then $C \subseteq A$, so either $\mu(C) = \{0\}$ or $C = A$, whence $C = B$.

Example 3.6 (i) Suppose that $\mu_1, \mu_2 : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ are two monotone set multifunctions such that $\mu_1(\emptyset) = \mu_2(\emptyset) = \{0\}$ and $\mu_1(A) \subseteq \mu_2(A)$, for every $A \in \mathcal{C}$ (for instance, one can think to $\mu_1, \mu_2 : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, $\mu_1(A) = [0, m_1(A)], \mu_2(A) = [0, m_2(A)]$ for every $A \in \mathcal{C}, m_1, m_2 : \mathcal{C} \to \mathbb{R}_+$ being monotone, $m_1(A) \leq m_2(A)$, for every $A \in \mathcal{C}$, $m_1(\emptyset) = m_2(\emptyset) = 0$). Then any minimal atom of μ_2 is a minimal atom of μ_1.

(ii) Let be $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}), \mu(A) = [-m_1(A), m_2(A)]$ for every $A \in \mathcal{C}$, where $m_1, m_2 : \mathcal{C} \to \mathbb{R}_+$, $m_1(\emptyset) = m_2(\emptyset) = 0$. Then a set $A \in \mathcal{C}$ is a minimal atom of μ iff A is a minimal atom for both m_1 and m_2 in the sense of (Mesiar *et al*., 2017; Ouyang *et al*., 2015).

(iii) If $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}), \mu(A) = \{m(A)\}$, for every $A \in \mathcal{C}$, where $m : \mathcal{C} \to \mathbb{R}_+, m(\emptyset) = 0$, then a set $A \in \mathcal{C}$ is a minimal atom of μ iff A is a minimal atom for m in the sense of (Mesiar *et al*., 2017; Ouyang *et al*., 2015).

In this way, one sees that Definition 3.1 - (i) generalizes to the set valued case the corresponding notion introduced by (Mesiar *et al*., 2017; Ouyang *et al*., 2015).

Definition 3.7 (i) *If* $\mu : \mathcal{C} \to \mathcal{P}_f(X)$, *let be the variation of* μ, $\overline{\mu} : \mathcal{P}(T) \to [0, \infty]$, *which is defined for every* $A \in \mathcal{P}(T)$ *by:*

$$\overline{\mu}(A) = \sup\left\{ \sum_{i=1}^{p} | \mu(A_i) |; A = \bigcup_{i=1}^{p} A_i, A_i \in \mathcal{X}, \forall i = \overline{1, p}, A_i \cap A_j = \emptyset, i \neq j \right\}.$$

(ii) *We say that* μ *is of finite variation if* $\overline{\mu}(T) < \infty$.

Remark 3.8 For every $A \in \mathcal{C}$, we have $\overline{\mu}(A) \geq |\mu(A)|$. Consequently, if $A \in \mathcal{C}$ is a minimal atom of $\overline{\mu}$ in the sense of (Mesiar *et al*., 2017; Ouyang *et al*., 2015), then A is a minimal atom of μ.

Conversely, if $A \in \mathcal{C}$ is a minimal atom of μ, then it is also an atom of μ, so $\overline{\mu}(A) \geq |\mu(A)|$, whence A is a minimal atom of $\overline{\mu}$.

Remark 3.9 (i) Any set $A \in \mathcal{C}$ that can be written as $\overset{p}{\underset{i=1}{\cup}} A_i$ (where for every $i = \overline{1, p}$, $A_i \in \mathcal{C}$ are different minimal atoms of μ), is partitioned in this way, since by Proposition 3.3 we have $A_i \cap A_j = \emptyset$, $i \neq j$.

Since any minimal atom is an atom, then in this case $\overline{\mu}(A_i) = |\mu(A_i)|$, for every $i = \overline{1, p}$. Consequently, if, moreover, μ is a multisubmeasure of finite variation in the sense of (Gavriluţ, 2009), then by (Gavriluţ, 2009), $\overline{\mu}$ is finitely additive, so $\overline{\mu}(A) = \sum_{i=1}^{p} |\mu(A_i)|$.

(ii) (non-decomposability of minimal atoms) Any minimal atom $A \in \mathcal{C}$ can not be partitioned (its only partition is $\{A, \emptyset\}$).

The converse of the last statement also holds:

Proposition 3.10 *Any non-partitionable atom* $A \in \mathcal{C}$ *is a minimal atom.*

Proof. Since A is an atom, then $\mu(A) \supseteq \{0\}, \mu(A) \neq \{0\}$. On the other hand, because A is non-partitionable, there can not exist two nonvoid disjoint subsets of A, let us say $A_1, A_2 \in \mathcal{C}$.

Let be now arbitrary $B \in \mathcal{C}$, with $B \subseteq A$. One has either $\mu(B) = \{0\}$ or $\mu(B) \supseteq \{0\}, \mu(B) \neq \{0\}$. In the latter situation, we can have only $B = A$ (if not, $\{A \setminus B, B\}$ is a partition of A, which is a contradiction).

Corollary 3.11 *An atom is minimal if and only if it is not partitionable.*

Theorem 3.12 *If* T *is finite,* μ *is null-additive and* $\{A_i\}_{i=\overline{1,p}}$ *is the set of all minimal different atoms contained in a set* $A \in \mathcal{C}$, *with* $\mu(A) \supseteq \{0\}, \mu(A) \neq \{0\}$, *then* $\mu(A) = \mu(\overset{p}{\underset{i=1}{\cup}} A_i)$ *(so, the minimal atoms are the only ones which are important from the "measurement" point of view).*

Proof. $\mu(A \setminus \overset{p}{\underset{i=1}{\cup}} A_i) = \{0\}$ (if not, there exists another minimal atom of μ). By the null-additivity of μ, one gets $\mu(A) = \mu(\overset{p}{\underset{i=1}{\cup}} A_i)$.

4. From the Standard Mathematical Atom to the Fractal Atom by Means of a Physical Procedure

Let T be an abstract nonvoid set, $\mathcal{C}$ a lattice of subsets of T and $m : \mathcal{C} \to \mathbb{R}_+$ an arbitrary set function with $m(\varnothing) = 0$. One can immediately generalize the notions of a pseudo-atom / minimal atom, respectively, to the case when $\mathcal{C}$ is only a lattice and not necessarily a ring.

Example 4.1 (i) It T is a nonempty metric space, then the Hausdorff dimension $\dim_{Haus} : \mathcal{P}(T) \to \mathbb{R}$ (Mandelbrot, 1983) is a monotone real function. Evidently, $\dim_{Haus}(\varnothing) = 0$.

(ii) For every $d \geq 0$, the Hausdorff measure $H^d : \mathcal{P}(T) \to \mathbb{R}$ is an outer measure, so, particularly, it is a submeasure.

Remark 4.2 (i) The union of two sets A and B having the fractal dimensions D_A, respectively, D_B has the fractal dimension $D_{A \cup B} = \max\{D_A, D_B\}$;

(ii) The intersection of two sets A and B having the fractal dimensions D_A, respectively, D_B has the fractal dimension $D_{A \cap B} = D_A + D_B - d$, where d is the embedding Euclidean dimension (Iannaccone and Khokha, 1995).

The following definition is then consistent:

Definition 4.3 *A pseudo-atom / minimal atom, respectively,* $A \in \mathcal{C}$ *of* m *having the fractal dimension* D_A *is said to be a fractal pseudo-atom / fractal minimal atom, respectively*.

One can easily verify the following:

Proposition 4.4 *If* $A, B \in \mathcal{C}$ *are fractal pseudo-atoms of* m *and if* $m(A \cap B) > 0$, *then* $A \cap B$ *is a fractal pseudo-atom of* m *and* $m(A \cap B) = m(A) = m(B)$.

5. Concluding Remarks

The main conclusions of the present paper are the following:

i) Minimal atomicity in correspondence with Quantum Measure Theory is discussed and some physical applications are provided;

ii) The concept of atomicity (and, particularly, of minimal atomicity) is extended in the form of fractal atomicity, respectively, fractal minimal atomicity. Some mathematical properties of fractal minimal atomicity are given.

REFERENCES

Cavaliere P., Ventriglia F., *On Nonatomicity for Non-Additive Functions*, Journal of Mathematical Analysis and Applications, 415, *1*, 358-372 (2014).

Chiţescu I., *Finitely Purely Atomic Measures and* Λ^p *-Spaces*, An. Univ. Bucureşti, Şt. Natur., **24**, 23-29 (1975).

Chiţescu I., *Finitely Purely Atomic Measures: Coincidence and Rigidity Properties*, Rend. Circ. Mat. Palermo (2) 50, *3*, 455-476 (2001).

Gavriluţ A., Agop M., *An Introduction to the Mathematical World of Atomicity through a Physical Approach*, Ars Longa Publishing House, Iaşi, 2016.

Gavriluţ A., *Regular Set Multifunctions*, Pim Publishing House, Iaşi, 2012.

Gavriluţ A., Croitoru A., *Non-Atomicity for Fuzzy and Non-Fuzzy Multivalued Set Functions*, Fuzzy Sets and Systems, **160**, 2106-2116 (2009).

Gavriluţ A., *Fuzzy Gould Integrability on Atoms*, Iranian Journal of Fuzzy Systems, **8**, *3*, 113-124 (2011).

Gavriluţ A., Croitoru A., *On the Darboux Property in the Multivalued Case*, Annals of the University of Craiova, Math. Comp. Sci. Ser., **35**, 130-138 (2008).

Gavriluţ A., *Non-Atomicity and the Darboux Property for Fuzzy and Non-Fuzzy Borel/Baire Multivalued Set Functions*, Fuzzy Sets and Systems, **160**, 1308-1317 (2009), Erratum in Fuzzy Sets and Systems, **161**, 2612-2613 (2010).

Gavriluţ A., Croitoru A., *Pseudo-Atoms and Darboux Property for Set Multifunctions*, Fuzzy Sets and Systems, **161**, *22*, 2897-2908 (2010).

Gavriluţ A., Iosif A., Croitoru A., *The Gould Integral in Banach Lattices*, Positivity, **19**, *1*, 65-82 (2015).

Gudder S., *Quantum Measure Theory*, Mathematica Slovaca, **60**, 681-700 (2010).

Gudder S., *Quantum Measure and Integration Theory*, J. Math. Phys., **50**, 123509 (2009a).

Gudder S., *Quantum Integrals and Anhomomorphic Logics*, arXiv: quant-ph (0911.1572), 2009b.

Gudder S., *Quantum Measures and the Coevent Interpretation*, Rep. Math. Phys., 67, 137-156 (2011a).

Gudder S., *Quantum Measures and Integrals*, Mathematics Preprint Series, **39** (2011b), https://digitalcommons.du.edu/math_preprints/39.

Hartle J.B., *The Quantum Mechanics of Cosmology*, 1989, Lectures at Winter School on Quantum Cosmology and Baby Universes, Jerusalem, Israel, Dec 27, 1989 - Jan 4, 1990.

Hartle J.B., *Spacetime Quantum Mechanics and the Quantum Mechanics of Spacetime*, in Proceedings of the Les Houches Summer School on Gravitation and Quantizations, Les Houches, France, 6 Jul - 1 Aug 1992, J. Zinn-Justin, and B. Julia (Eds.), North-Holland (1995), arXiv: gr-qc/9304006.

Iannaccone P.M., Khokha M., *Fractal Geometry in Biological Systems: An Analitical Approach*, 1995.

Khare M., Singh A.K., *Atoms and Dobrakov Submeasures in Effect Algebras*, Fuzzy Sets and Systems, **159**, *9*, 1123-1128 (2008).

Li J., Mesiar R., Pap E., *Atoms of Weakly Null-Additive Monotone Measures and Integrals*, Information Sciences, 134-139 (2014).

Li J., Mesiar R., Pap E., Klement E.P., *Convergence Theorems for Monotone Measures*, Fuzzy Sets and Systems, **281**, 2015, 103-127.

Mandelbrot B.B., *The Fractal Geometry of Nature* (Updated and Augm. Ed.), W.H. Freeman, New York, 1983.

Mercheş I., Agop M., *Differentiability and Fractality in Dynamics of Physical Systems*, World Scientific, 2015.

Mesiar R., Li J., Ouyang Y., *On the Equality of Integrals,* Information Sciences, **393**, 82-90 (2017).

Michel O.D., Thomas B.G., *Mathematical Modeling for Complex Fluids and Flows*, Springer, New York, 2012.

Mitchell M., *Complexity: A Guided Tour*, Oxford University Press, Oxford, 2009.

Nottale L., *Scale Relativity and Fractal Space-Time. A New Approach to Unifying Relativity and Quantum Mechanics*, Imperial College Press, London, 2011.

Ouyang Y., Li J., Mesiar R., *Relationship between the Concave Integrals and the Pan-Integrals on Finite Spaces*, J. Math. Anal. Appl., **424**, 975-987 (2015).

Pap E., *The Range of Null-Additive Fuzzy and Non-Fuzzy Measures*, Fuzzy Sets and Systems, **65**, *1*, 105-115 (1994).

Pap E., *Null-Additive Set Functions*, Springer, Series: Mathematics and its Applications, **337** (1995).

Pap E., *Some Elements of the Classical Measure Theory* (Chapter 2 in Handbook of Measure Theory), 2002, 27-82.

Pap E., Gavriluţ A., Agop M., *Atomicity via Regularity for Non-Additive Set Multifunctions*, Soft Computing (Foundations), 2016, 1-6, DOI: 10.1007/s00500-015-2021-x.

Rao K.P.S.B., Rao M.B., *Theory of Charges*, Academic Press, Inc., New York, 1983.

Salgado R., *Some Identities for the q-Measure and its Generalizations*, Modern Phys. Lett. A 17 (2002), 711-728.

Schweizer B., Sklar A., *Probabilistic Metric Spaces*, Elsevier Science Publishing Co., Inc. Republished in 2005 by Dover Publications, Inc., with a New Preface, Errata, Notes, and Supplementary References (1983).

Sorkin R.D., *Quantum Mechanics as Quantum Measure Theory*, Mod. Phys. Lett. A9 3119-3128 (1994) arXiv: gr-qc/9401003.

Sorkin R.D., *Quantum Dynamics without the Wave Function*, J. Phys. A: Math. Theor., **40**, 3207-3231 (2007), arXiv:quant-ph/0610204.

Sorkin R.D., *Quantum Measure Theory and its Interpretation*, in Quantum Classical Correspondence: Proceedings of the 4^{th} Drexel Symposium on Quantum Non-Integrability, pag. 229-251, International Press, Cambridge Mass., 1997, D.H. Feng and B.-L. Hu (Editors), arXiv:gr-qc/9507057**.**

Surya S., Waldlden P., *Quantum Covers in q-Measure Theory* (2008), arXiv: quant-ph 0809.1951.

Suzuki H., *Atoms of Fuzzy Measures and Fuzzy Integrals*, Fuzzy Sets and Systems, **41**, 329-342 (1991).
Wu C., Bo S., *Pseudo-Atoms of Fuzzy and Non-Fuzzy Measures*, Fuzzy Sets and Systems, **158**, 1258-1272 (2007).

VARIANTE ALE ATOMICITĂŢII ŞI UNELE APLICAŢII FIZICE

(Rezumat)

În această lucrare, prezentăm unele rezultate referitoare la diferite forme de atomicitate din perspectiva teoriei măsurii cuantice şi stabilim câteva aplicaţii în fizică. Mai precis, extindem conceptul matematic de atomicitate minimală şi, pe baza remarcii conform căreia mecanica cuantică este un caz particular de mecanică fractală la o rezoluţie de scală specifică, introducem conceptul de atomicitate fractală (şi, în particular, cel de atomicitate minimală fractală). De asemenea, indicăm unele proprietăţi matematice ale acestora.