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FIXED POINT THEOREMS FOR TWO PAIRS OF MAPPINGS SATISFYING A NEW TYPE OF COMMON LIMIT RANGE PROPERTY IN *G* – METRIC SPACES

BY

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Abstract. In this paper we introduce a new type of common limit range property which generalize the known definition from (Imdad *et al.*, 2012). We obtain some generalizations of main results proved in (Giniswamy and Maheshwari, 2014; Popa and Patriciu, 2014; Popa and Patriciu, 2016) in *G* - metric space. As applications, some fixed point results for two pairs of mappings satisfying contractive conditions of integral type and for ϕ - contractive mappings in *G* - metric spaces are obtained.

Keywords: *G* - metric space; fixed point; almost altering distance; common limit range property; implicit relation.

1. Introduction and Preliminaries

The concept of compatible self mappings is often used in fixed point theory to prove existence theorems and it was introduced by Jungck (1986).

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Let (X,d) be a metric space. Two self mappings of X, S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Let f, g be self mappings of a nonempty set X. A point $x \in X$ is a coincidence point of f and g if $w = fx = gx$ and w is said to be a point of coincidence of f and g.

The set of all coincidence points of f and g is denoted by $C(f,g)$.

In (Jungck, 1996) Jungck introduced the notion of weakly compatible mappings.

Definition 1.1 (Jungck, 1996) *Let X be a nonempty set and $f, g : X \to X$. f and g are weakly compatible if $fgu = gfu$ for all $u \in C(f,g)$.*

In 2011, Sintunavarat and Kumam (Sintunavarat and Kumam, 2011) introduced the notion of common limit range property.

Definition 1.2 (Sintunavarat and Kumam, 2011) *A pair (A,S) of self mappings of a metric space (X,d) is said to satisfy common limit range property with respect to S, denoted $CLR_{(S)}$ – property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in S(X)$.*

Thus we can infer that a pair (A,S) satisfying $(E.A)$ - property, along with the closedness of the subspace $S(X)$, always has $CLR_{(S)}$ – property.

Recently, Imdad *et al*. (2012) extended the notion of common limit range property to two pairs of self mappings.

Definition 1.3 (Imdad *et al.*, 2012) *Two pairs (A,S) and (B,T) of self mappings of a metric space (X,d) are said to satisfy common limit range property with respect to S and T, denoted $CLR_{(S,T)}$ – property, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$, for some $t \in S(X) \cap T(X)$.*

Some results for pairs of mappings satisfying $CLR_{(S)}$ – and $CLR_{(S,T)}$ – property are obtained in (Imdad and Chauhan, 2013; Imdad *et al.*, 2013; Imdad *et al.*, 2014; Giniswamy and Maheshwari, 2014) and in other papers.

Popa (2017) introduced a new type of common limit range property.

Definition 1.4 *Let A, S, T be self mappings of a metric space (X,d). The pair (A,S) is said to satisfy common limit range property with respect to T, denoted $CLR_{(A,S)T}$ – property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in S(X) \cap T(X)$.*

Example 1.1 *Let* $\mathbb{R}_+$ *be the metric space with the usual metric,* $Ax = \frac{x^2+1}{2}$, $Sx = \frac{x+1}{2}$, $Tx = x + \frac{1}{4}$. *Then* $S(X) = \left[\frac{1}{2}, \infty\right)$, $T(X) = \left[\frac{1}{4}, \infty\right)$, $S(X) \cap T(X) = \left[\frac{1}{2}, \infty\right)$. *Let* $\{x_n\}$ *be a sequence with* $\lim_{n\to\infty} x_n = 0$. *Then,*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{2} = z, z \in S(X) \cap T(X).$$

Remark 1.1 (Popa, 2017) *Let* A, B, S *and* T *be self mappings of a metric space* (X, d). *If* (A, S) *and* (B, T) *satisfy* $CLR_{(S,T)}$ *– property, then* (A, S) *satisfy* $CLR_{(A,S)T}$ *–property.*

The purpose of this paper is to prove a general fixed point theorem for mappings with $CLR_{(A,S)T}$ – property and satisfying an implicit relation. As applications, some fixed point results for two pairs of mappings satisfying contractive condition of integral type and for φ - contractive mappings in G - metric spaces are obtained.

2. Preliminaries

In (Dhage, 1992; Dhage, 2000), Dhage introduced a new class of generalized metric spaces, called D - metric space. Mustafa and Sims (2003, 2006) proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G - metric space.

In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G - metric spaces under certain conditions (Mustafa *et al.*, 2008; Shatanawi, 2010; Popa and Patriciu, 2012) and other papers.

Definition 2.1 (Mustafa and Sims, 2006) *Let X be a nonempty set and* $G : X^3 \to \mathbb{R}_+$ *be a function satisfying the following properties:*

$(G_1) : G(x, y, z) = 0$ *for* $x = y = z$,

$(G_2) : 0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,

$(G_3) : G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$,

$(G_4) : G(x, y, z) = G(y, z, x) = \ldots$ *(symmetry in all three variables),*

$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(triangle inequality).*

The function G is called a G - metric on X and (X, G) *is called a G - metric space.*

Remark 2.1 *Let* (X, G) *be a G - metric space. If* $y = z$, *then the mapping* $(x, y) \to G(x, y, y)$ *is a quasi - metric on X. Hence,* (X, Q), *where*

$Q(x,y)=G(x,y,y)$ is a quasi - metric and since every metric space is a particular case of quasi - metric space it follows that the notion of G - metric space is a generalization of a metric space.

Definition 2.2 (Mustafa and Sims, 2006) *Let (X,G) be a G - metric space. A sequence $\{x_n\}$ in X is said to be*

a) *G – convergent to $x \in X$ if for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $m,n \geq k$ we have $G(x,x_n,x_m) < \varepsilon$.*

b) *G - Cauchy if for $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $m,n,p \geq k$, $G(x_n,x_m,x_p) < \varepsilon$, that is $G(x_n,x_m,x_p) \to 0$ as $m,n,p \to \infty$.*

A G - metric space (X,G) is said to be G - complete if every G - Cauchy sequence is G - convergent.

Lemma 2.1 (Mustafa and Sims**,** 2006) *Let (X,G) be a G - metric space. Then, the following conditions are equivalent:*

1) *$\{x_n\}$ is G - convergent to x;*
2) *$G(x_n,x_n,x) \to 0$ as $n \to \infty$;*
3) *$G(x_n,x,x) \to 0$ as $n \to \infty$;*
4) *$G(x_n,x_m,x) \to 0$ as $n,m \to \infty$.*

Lemma 2.2 (Mustafa and Sims**,** 2006) *If (X,G) is a G - metric space, then the following conditions are equivalent:*

1) *$\{x_n\}$ is G - Cauchy;*
2) *for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$ for all $m,n \in \mathbb{N}$ with $m,n \geq k$.*

Quite recently, in (Popa and Patriciu, 2016) a general fixed point theorem for two pairs of mappings satisfying $CLR_{(S,T)}$ – property is proved.

Remark 2.2. A similar definition with Definition 1.4 we have in G - metric spaces.

Definition 2.3 (Mustafa and Sims, 2006) *A G - metric is symmetric if $G(x,x,y)=G(y,x,x)$ for all $x,y \in X$.*

Definition 2.4 (Khan *et al.*, 1984) *An altering distance is a function $\phi:[0,\infty) \to [0,\infty)$ satisfying:*

(ϕ_1): ϕ is increasing and continuous;

(ϕ_2): $\phi(t)=0$ if and only if $t=0$.

Fixed point theorems involving altering distance have been studied in (Popa and Mocanu, 2009; Popa and Patriciu, 2014; Sastri and Babu, 1998; Sastri and Babu, 1999), and in other papers.

Definition 2.5 *A function $\psi:[0,\infty) \to [0,\infty)$ is an almost altering distance if:*

$(\psi_1):\psi$ *is continuous;*

$(\psi_2):\psi(t)=0$ *if and only if* $t=0$.

Remark 2.3 (Popa and Patriciu, 2016) *Every altering distance is an almost altering distance, but the converse is not true.*

3. Implicit Relations

Various fixed point theorems and common fixed point theorems have been unified considering a general contractive condition defined by an implicit relation in (Popa, 1997; Popa, 1999) and in other papers. The study of fixed points for mappings satisfying implicit relations has been initiated in (Popa and Patriciu, 2012; Popa and Patriciu, 2013) in the setting of G - metric spaces, the case of pairs of mappings with common limit range property beeing was studied first in (Imdad and Chauhan, 2013) in the setting of metric spaces, then in the setting of G - metric spaces in (Popa and Patriciu, 2014).

A new class of implicit relations was introduced in 2008 by Ali and Imdad (Ali and Imdad, 2008).

Definition 3.1 (Ali and Imdad, 2008) *Let* F_{AI} *be the family of lower semi - continuous functions* $F(t_1,...,t_6):\mathbb{R}_+^6\to\mathbb{R}$ *satisfying the following conditions:*

$(F_1): F(t,0,t,0,0,t)>0$ *for all* $t>0$,

$(F_2): F(t,0,0,t,t,0)>0$ *for all* $t>0$,

$(F_3): F(t,t,0,0,t,t)>0$ *for all* $t>0$.

Example 3.1 $F(t_1,...,t_6)=t_1-at_2-bt_3-ct_4-dt_5-et_6$ **,** *where* $a,b,c,d,e\geq 0$ *and* $a+b+c+d+e<1$.

Example 3.2 $F(t_1,...,t_6)=t_1-k\max\left\{t_2,t_3,t_4,\dfrac{t_5+t_6}{2}\right\}$, *where* $k\in[0,1)$.

Example 3.3 $F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,...,t_6\}$**,** *where* $k\in[0,1)$.

Example 3.4 $F(t_1,...,t_6)=t_1-k\max\left\{t_2,\dfrac{t_3+t_4}{2},\dfrac{t_5+t_6}{2}\right\}$, *where* $k\in[0,1)$.

Example 3.5 $F(t_1,...,t_6)=t_1-at_2-b\max\{t_3,t_4\}-c\max\{t_2,t_5,t_6\}$, *where* $a,b,c\geq 0$ *and* $a+b+c<1$.

Example 3.6 $F(t_1,...,t_6)=t_1-\alpha\max\{t_2,t_3,t_4\}-(1-\alpha)(at_5+bt_6)$, *where* $\alpha\in(0,1)$, $a,b\geq 0$ *and* $a+b<1$.

Example 3.7 $F(t_1,...,t_6) = t_1 - at_2 - \frac{b(t_5 + t_6)}{1 + t_3 + t_4}$, *where* $a, b \geq 0$ *and* $a + 2b < 1$.

Example 3.8 $F(t_1,...,t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $a, b, c \geq 0$ and $a + b + c < 1$.

For other examples, see (Ali and Imdad, 2008).

The following theorem is proved in (Popa and Patriciu, 2016).

Theorem 3.1 (Popa and Patriciu, 2016) *Let A, B, S and T be self mappings of a G - metric space* (X, G) *satisfying inequality*

$$F\begin{pmatrix} \psi(G(Ax, By, By)), \psi(G(Sx, Ty, Ty)), \psi(G(Sx, Sx, Ax)), \\ \psi(G(Ty, By, By)), \psi(G(Sx, By, By)), \psi(G(Ax, Ty, Ty)) \end{pmatrix} \leq 0 \tag{3.1}$$

for all $x, y \in X$, $F \in F_{AI}$ *and* ψ *is an almost altering distance.*

If (A, S) *and* (B, T) *satisfy* $CLR_{(S,T)}$ *– property, then*

1) $C(A, S) \neq \emptyset$,

2) $C(B, T) \neq \emptyset$.

Moreover, if (A, S) *and* (B, T) *are weakly compatible, then A, B, S and T have a unique common fixed point.*

This theorem unifies the results from (Giniswamy and Maheshwari, 2014) and generalizes the main results from (Popa and Patriciu, 2014).

4. Main Results

Lemma 4.1 (Abbas and Rhoades, 2009) *Let f,g be two weakly compatible self mappings of a nonempty set X. If f and g have a unique point of coincidence* $w = fx = gx$ *for some* $x \in X$, *then w is the unique common fixed point of f and g.*

Theorem 4.1 (Popa and Patriciu, 2016) *Let A, B, S and T be self mappings of a G - metric space* (X, G) *satisfying the inequality*

$$F\begin{pmatrix} \psi(G(Ax, By, By)), \psi(G(Sx, Ty, Ty)), \psi(G(Sx, Sx, Ax)), \\ \psi(G(Ty, By, By)), \psi(G(Sx, By, By)), \psi(G(Ax, Ty, Ty)) \end{pmatrix} \leq 0 \tag{4.1}$$

for all $x, y \in X$, *where* $F \in F_{AI}$ *and* ψ *is an almost altering distance.*

If there exist $u, v \in X$ *such that* $Au = Su$ *and* $Tv = Bv$, *then there exists* $t \in X$ *such that t is the unique point of coincidence of A and S, as well t is the unique point of coincidence of B and T.*

Theorem 4.2 *Let A, B, S and T be self mappings of a G - metric space* (X, G) *satisfying the inequality (4.1) for all* $x, y \in X$, *where* $F \in F_{AI}$ *and* ψ *is an almost altering distance.*

Assume that (A,S) *and T satisfy* $CLR_{(A,S)T}$ − *property. Then*

i) $C(A,S) \neq \emptyset$,

ii) $C(B,T) \neq \emptyset$.

Moreover, if in addition (A,S) *and* (B,T) *are weakly compatible, then A, B, S and T have a unique common fixed point.*

Proof. Since (A,S) and T satisfy $CLR_{(A,S)T}$ − property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \text{ and } z \in S(X) \cap T(X).$$

Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$. By (4.1) we obtain

$$F\begin{pmatrix} \psi(G(Ax_n, Bu, Bu)), \psi(G(Sx_n, Tu, Tu)), \psi(G(Sx_n, Sx_n, Ax_n)), \\ \psi(G(Tu, Bu, Bu)), \psi(G(Sx_n, Bu, Bu)), \psi(G(Ax_n, Tu, Tu)) \end{pmatrix} \leq 0.$$

Letting n tend to infinity we obtain

$$F(\psi(G(z, Bu, Bu)), 0, 0, \psi(G(z, Bu, Bu)), \psi(G(z, Bu, Bu)), 0)) \leq 0,$$

a contradiction of (F_2) if $\psi(G(z, Bu, Bu)) > 0$. Hence, $G(z, Bu, Bu) = 0$, which implies $z = Bu = Tu$ and $C(B,T) \neq \emptyset$.

Since $z \in S(X)$, there exists $v \in X$ such that $z = Sv$. By (4.1) we obtain

$$F\begin{pmatrix} \psi(G(Av, Bu, Bu)), \psi(G(Sv, Tu, Tu)), \psi(G(Sv, Sv, Av)), \\ \psi(G(Tu, Bu, Bu)), \psi(G(Sv, Bu, Bu)), \psi(G(Av, Tu, Tu)) \end{pmatrix} \leq 0,$$

i.e.

$$F(\psi(G(Av, z, z)), 0, \psi(G(z, z, Av)), 0, 0, \psi(G(Av, z, z))) \leq 0.$$

a contradiction of (F_1) if $\psi(G(Av, z, z))) > 0$. Hence, $\psi(G(Av, z, z))) = 0$, which implies $z = Av = Sv$ and $C(A,S) \neq \emptyset$ and z is a point of coincidence of A and S. Hence z is a common fixed point of coincidence of (A,S) and (B,T).

By Theorem 4.1, z is the unique point of coincidence of (A,S) and (B,T).

Moreover, if in addition (A,S) and (B,T) are weakly compatible, then by Lemma 4.1, z is the unique fixed point of A, B, S and T.

Remark 4.1 *If* A, B, S and T *have* $CLR_{(S,T)}$ − *property, then by Theorem 4.2 and Remark 1.2, we obtain Theorem 3.1.*

If $\psi(t) = t$, then by Theorem 4.2 we obtain

Theorem 4.3 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *satisfying the inequality*

$$F\begin{pmatrix} G(Ax,By,By), G(Sx,Ty,Ty), G(Sx,Sx,Ax), \\ G(Ty,By,By), G(Sx,By,By), G(Ax,Ty,Ty) \end{pmatrix} \leq 0 \tag{4.2}$$

for all $x, y \in X$ *, where* $F \in F_{AI}$.

Assume that (A,S) *and* T *satisfy* $CLR_{(A,S)T}$ *– property. Then*

i) $C(A,S) \neq \emptyset$,

ii) $C(B,T) \neq \emptyset$.

Moreover, if in addition (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Corollary 4.1 (**Theorem 2.5** (Giniswamy and Maheshwari, 2014)) *Let* (X,G) *be a G - metric space and A,B,S and T be self mappings of* X *such that*

1) (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *– property,*

2)
$$G(Ax,By,Bz) \leq pG(Sx,Ty,Ty) + qG(Sx,Sx,Ax) + \\ + rG(Ty,Bz,Bz) + t[G(Ax,Ty,Tz) + G(Sx,By,By)], \tag{4.3}$$

for all $x, y, z \in X$ *, where* $p,q,r,t \geq 0$ *and* $p+q+r+t < 1$.

If (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Proof. Let $y = z$, then by (4.3) we obtain a particular case of (4.2) and the proof it follows by Remark 1.1, Theorem 4.3 and Example 3.1.

Corollary 4.2 (**Theorem 2.6** (Giniswamy and Maheshwari, 2014)) *Let* (X,G) *be a G - metric space and A,B,S and T be self mappings of* X *such that*

1) (A,S) *and* (B,T) *satisfy* $CLR_{(S,T)}$ *– property,*

2) $G(Ax,By,Bz) \leq hu(x,y,z)$*, where* $h \in (0,1)$, $x,y,z \in X$ *and*

$$u(x,y,z) = \max\left\{ G(Sx,Ty,Tz), G(Ax,Sx,Sx), \\ G(Sx,Ty,Ty), \frac{G(Ax,Ty,Tz) + G(Sx,By,Bz)}{2} \right\}. \tag{4.4}$$

Then (A,S) *and* (B,T) *have a point of coincidence in* X *. Moreover, if in addition* (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Proof. Let $y = z$, then by (4.4) we obtain

$$G(Ax,By,By) \leq h \max\left\{ G(Sx,Ty,Ty), G(Sx,Sx,Ax), \\ G(Ty,By,By), \frac{G(Sx,By,By) + G(Ax,Ty,Ty)}{2} \right\}.$$

The proof it follows by Remark 1.1, Example 3.2 and Theorem 4.2.

For a function $f:(X,G)\to(X,G)$ we denote

$$Fix(f)=\{x\in X : x=fx\}.$$

Theorem 4.4 (**Theorem 4.7** (Popa and Patriciu, 2016)) *Let A,B,S and T be self mappings of a G - metric space (X,G). If the inequality (4.1) holds for all $x,y\in X$, where $F\in F_{AI}$ and ψ is an almost altering distance, then*

$$[Fix(S)\cap Fix(T)]\cap Fix(A)=[Fix(S)\cap Fix(T)]\cap Fix(B).$$

Theorems 4.2 and 4.4 imply the following one.

Theorem 4.5 *Let S,T and $\{A_i\}_{i\in\mathbb{N}^*}$ be self mappings of a G - metric space (X,G) satisfying the inequality*

$$F\begin{pmatrix} \psi(G(A_i x, A_{i+1}y, A_{i+1}y)), \psi(G(Sx,Ty,Ty)), \psi(G(Sx,Sx,A_i x)), \\ \psi(G(Ty, A_{i+1}y, A_{i+1}y)), \psi(G(Sx, A_{i+1}y, A_{i+1}y)), \psi(G(A_i x, Ty, Ty)) \end{pmatrix}\leq 0, \quad (4.5)$$

for all $x,y\in X$, $i\in\mathbb{N}^*$, where $F\in F_{AI}$ and ψ is an almost altering distance.

If (A_1,S) and T satisfy $CLR_{(A_1,S)T}$ − property and (A_1,S) and (A_2,T) are weakly compatible, then S,T and $\{A_i\}_{i\in\mathbb{N}^*}$ have.a unique common fixed point.

Proof. Let $i=1$. By Theorem 4.2, A_1,A_2,S and T have a unique common fixed point z. Then we have $z\in[Fix(S)\cap Fix(T)]\cap Fix(A_1)$. Suppose that there exists an other point z_1 such that $z\in[Fix(S)\cap Fix(T)]\cap Fix(A_1)$. By the first part of the proof of Theorem 4.4, $z_1\in Fix(A_2)$. Hence z_1 is an other common fixed point of A_1,A_2,S and T, a contradiction of Theorem 4.2, hence, $z\in[Fix(S)\cap Fix(T)]\cap Fix(A_1)$.

Let $i=2$. By Theorem 4.4,

$$z\in[Fix(S)\cap Fix(T)]\cap Fix(A_3)=\ldots=[Fix(S)\cap Fix(T)]\cap Fix(A_k)=\ldots.$$

Hence, z is the unique common fixed point of S,T and $\{A_i\}_{i\in\mathbb{N}^*}$.

If $\psi(t)=t$, from Theorem 4.5 we obtain

Theorem 4.6 *Let S,T and $\{A_i\}_{i\in\mathbb{N}^*}$ be self mappings of a G - metric space (X,G) satisfying the inequality*

$$F\begin{pmatrix} G(A_i x, A_{i+1}y, A_{i+1}y), G(Sx,Ty,Ty), G(Sx,Sx,A_i x), \\ G(Ty, A_{i+1}y, A_{i+1}y), G(Sx, A_{i+1}y, A_{i+1}y), G(A_i x, Ty, Ty) \end{pmatrix}\leq 0, \quad (4.6)$$

for all $x,y\in X$, where $i\in\mathbb{N}^*$ and $F\in F_{AI}$.

Assume that (A_1, S) *and* T *satisfy* $CLR_{(A_1,S)T}$ *– property and* (A_1, S) *and* (A_2, T) *are weakly compatible. Then* S, T *and* $\{A_i\}_{i\in\mathbb{N}^*}$ *have a unique common fixed point.*

5. Applications

5.1. Fixed Points for Mappings Satisfying Contractive Conditions of Integral Type in *G* - Metric Spaces

In (Branciari, 2002), Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 5.1 (Branciari, 2002) *Let* (X, d) *be a complete metric space,* $c \in (0,1)$ *and* $f : X \to X$ *such that for all* $x, y \in X$

$$\int_0^{d(fx,fy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt$$

whenever $h:[0,\infty)\to[0,\infty)$ *is a Lebesgue measurable mapping which is summable (i.e., with finite integral) on each compact subset of* $[0,\infty)$, *such that* $\int_0^{\varepsilon} h(t)dt > 0$, *for each* $\varepsilon > 0$. *Then,* f *has a unique fixed point* $z \in X$ *such that for all* $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in (Popa and Mocanu, 2007; Popa and Mocanu, 2009).

Lemma 5.1 *Let* $h:[0,\infty)\to[0,\infty)$ *as in Theorem 5.1. Then* $\psi:[0,\infty)\to[0,\infty)$ *defined by* $\psi(t) = \int_0^t h(x)dx$ *is an altering distance, in particular is an almost altering distance.*

Proof. The proof it follows from Lemma 2.5 (Popa and Mocanu, 2009).

Theorem 5.2 *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *such that*

$$F\begin{pmatrix} \int_0^{G(Ax,By,By)} h(t)dt, & \int_0^{G(Sx,Ty,Ty)} h(t)dt, & \int_0^{G(Sx,Sx,Ax)} h(t)dt, \\ \int_0^{G(Ty,By,By)} h(t)dt, & \int_0^{G(Sx,By,By)} h(t)dt, & \int_0^{G(Ax,Ty,Ty)} h(t)dt \end{pmatrix} \le 0, \qquad (5.1)$$

for all $x, y \in X$ *, where* $F \in F_{AI}$ *and* $h(t)$ *as in Theorem 5.1.*

Assume that (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *– property. Then*

i) $C(A, S) \neq \emptyset$,

ii) $C(B, T) \neq \emptyset$.

Moreover, if in addition (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. By Lemma 5.1, $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance. By (5.1) we obtain

$$F\begin{pmatrix} \psi(G(Ax, By, By)), \psi(G(Sx, Ty, Ty)), \psi(G(Sx, Sx, Ax)), \\ \psi(G(Ty, By, By)), \psi(G(Sx, By, By)), \psi(G(Ax, Ty, Ty)) \end{pmatrix} \leq 0 .$$

Hence the conditions of Theorem 4.2 are satisfied and the conclusions of Theorem 5.3 follows.

From Theorem 5.2 and Example 3.2 we obtain

Theorem 5.3 *Let* (X, G) *be a* G *- metric space and* A, B, S *and* T *be self mappings of* X *satisfying*

$$\int_0^{G(Ax,By,By)} h(t)dt \leq k \max\left\{ \int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Sx,Sx,Ax)} h(t)dt, \int_0^{G(Ty,By,By)} h(t)dt, \int_0^{G(Sx,By,By)} h(t)dt, \int_0^{G(Ax,Ty,Ty)} h(t)dt \right\},$$

for all $x, y \in X$ *, where* $k \in [0,1)$ *and* $h(t)$ *is as in Theorem 5.1.*

Assume that (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *– property. Then*

i) $C(A, S) \neq \emptyset$,

ii) $C(B, T) \neq \emptyset$.

Moreover, if in addition (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Remark 5.1 *By Theorem 5.2 and Examples 3.2 – 3.8 we obtain new particular results.*

5.2. Fixed Points for Mappings Satisfying φ - Contractive Conditions in G - Metric Spaces

As in (Matkowski, 1997), let ϕ be the set of all real nondecreasing continuous functions $\varphi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \varphi^n(t) = 0$, for all $t \in [0, \infty)$.

If $\varphi \in \phi$, then

1) $\varphi(t) < t$ for all $t \in (0, \infty)$,

2) $\varphi(0) = 0$.

The following functions $F:\mathbb{R}_+^6 \to \mathbb{R}$ satisfy conditions $(F_1)-(F_3)$.

Example 5.1 $F(t_1,...,t_6)=t_1-\varphi(\max\{t_2,t_3,...,t_6\})$.

Example 5.2 $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{2}\right\}\right)$.

Example 5.3 $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right)$.

Example 5.4 $F(t_1,...,t_6)=t_1-\varphi\left(\max\left\{t_2,\sqrt{t_3t_4},\sqrt{t_3t_5},\sqrt{t_4t_6},\sqrt{t_5t_6}\right\}\right)$.

Example 5.5 $F(t_1,...,t_6)=t_1-\varphi(at_2+bt_3+ct_4+dt_5+et_6)$, *where* $a,b,c,d,e\geq 0$ *and* $a+b+c+d+e<1$.

Example 5.6 $F(t_1,...,t_6)=t_1-\varphi\left(at_2+\frac{b\sqrt{t_5t_6}}{1+t_3+t_4}\right)$, *where* $a,b\geq 0$ *and* $a+b<1$.

Example 5.7 $F(t_1,...,t_6)=t_1-\varphi\left(at_2+b\max\{t_3,t_4\}+c\max\left\{\frac{t_3+t_4}{2},\frac{t_5+t_6}{2}\right\}\right)$, *where* $a,b,c\geq 0$ *and* $a+b+c<1$.

Example 5.8 $F(t_1,...,t_6)=t_1-\varphi\left(at_2+b\max\left\{\frac{2t_4+t_5}{3},\frac{2t_4+t_6}{3},\frac{t_3+t_5+t_6}{3}\right\}\right)$, *where* $a,b\geq 0$ *and* $a+b<1$.

By Theorem 4.2 and Example 5.1 we obtain

Theorem 5.4 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$\psi(G(Ax,By,By))\leq\varphi(\max\{\psi(G(Sx,Ty,Ty)),\psi(G(Sx,Sx,Ax)),\\ \psi(G(Ty,By,By)),\psi(G(Sx,By,By)),\psi(G(Ax,Ty,Ty))\}),$$

for all $x,y\in X$, *where* $\varphi\in\phi$ *and* ψ *is an almost altering distance.*

Assume that (A,S) *and* T *satisfy* $CLR_{(A,S)T}-$ *property. Then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if in addition (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

If $\psi(t)=t$, from Theorem 5.4 we obtain

Theorem 5.5 *Let* A,B,S *and* T *be self mappings of a* G *- metric space* (X,G) *such that*

$$G(Ax,By,By)\le \varphi(\max\{G(Sx,Ty,Ty),G(Sx,Sx,Ax),$$
$$G(Ty,By,By),G(Sx,By,By),G(Ax,Ty,Ty)\}),$$

for all $x,y\in X$ *, where* $\varphi\in\phi$.

Assume that (A,S) *and* T *satisfy* $CLR_{(A,S)T}$ – *property. Then*

i) $C(A,S)\neq\emptyset$,

ii) $C(B,T)\neq\emptyset$.

Moreover, if in addition (A,S) *and* (B,T) *are weakly compatible, then* A,B,S *and* T *have a unique common fixed point.*

Corollary 5.1 (Giniswamy and Maheshwari, 2014) *Let* (X,G) *be a symmetric* G *- metric space and* A,B,S *and* T *be self mappings of* X *such that*

1) (A,S) *and* T *satisfy* $CLR_{(S,T)}$ *– property,*

2) $G(Ax,By,Bz)\le\phi(\max\{G(Sx,Ty,Tz),G(Sx,By,Bz),G(Ty,By,Bz),G(By,Ty,Tz)\})$,

for all $x,y\in X$ *, where* $\varphi\in\phi$.

3) (A,S) *and* (B,T) *are weakly compatible.*

Then A,B,S *and* T *have a unique common fixed point.*

Proof. If $y=z$ we have

$$G(Ax,By,By)\le\varphi(\max\{G(Sx,Ty,Ty),G(Sx,By,By),G(Ty,By,By),G(By,Ty,Ty)\}),$$

Since G is symmetric ($G(By,Ty,Ty)=G(Ty,By,By)$) and φ is nondecreasing, then

$$G(Ax,By,By)\le\phi(\max\{G(Sx,Ty,Ty),G(Sx,By,By),G(Ty,By,By)\})$$
$$\le\phi(\max\{G(Sx,Ty,Ty),G(Sx,Ax,Ax),G(Ty,By,By),G(Sx,By,By),G(Ax,Ty,Ty)\})$$

and by Theorem 5.5, A,B,S *and* T have a unique common fixed point.

Remark 5.2 *Similarly, by Examples 5.2 – 5.6 and Theorem 5.5 we obtain new particular results.*

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TEOREME DE PUNCT FIX PENTRU DOUĂ PERECHI DE FUNCŢII CARE SATISFAC UN NOU TIP DE PROPRIETATE A LIMITEI COMUNE ÎN SPAŢII *G* – METRICE

(Rezumat)

În această lucrare introducem un nou tip de proprietate a limitei comune, care generalizează definiţia cunoscută din (Imdad *et al.*, 2012). Obţinem câteva generalizări a principalelor rezultate demonstrate în (Giniswamy and Maheshwari, 2014; Popa and Patriciu, 2014; Popa and Patriciu, 2016) în spaţii *G* – metrice. Ca aplicaţii, sunt obţinute câteva rezultate de punct fix pentru două perechi de funcţii care satisfac condiţii contractive de tip integral şi pentru funcţii ϕ - contractive în spaţii *G* - metrice.