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SOME RESULTS ON BIRKHOFF WEAK INTEGRABILITY

BY

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Abstract. We present some properties of Birkhoff weak *m*-integrable vector functions with respect to a non-negative set function *m*. A comparison result with Birkhoff simple integral and a characterization in terms of strongly Birkhoff weak integrability are established.

Keywords: Birkhoff weak integral; Birkhoff simple integral; non-negative set function; non-additive integral.

1. Introduction

The theory of non-additive measures and integrals has intensively developed in the last decades due to numerous applications in artificial intelligence, data mining, computer science, potential theory, subjective evaluation or decision making.

There exist different extensions of integrals to non-additive case. In (Croitoru *et al*., 2017), we defined such an extension of the Birkhoff integral (Birkhoff, 1935), called the Birkhoff weak integral for vector functions with respect to a non-negative set function. Other types of non-linear integrals and

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their applications have been studied for instance in Boccuto and Sambucini, 2012; Candeloro and Sambucini, 2014; Candeloro *et al*., 2016a; Candeloro *et al*., 2016b; Cascales and Rodriguez, 2005; Croitoru and Gavriluț, 2015; Croitoru *et al*., 2015, Croitoru *et al*., 2016; Croitoru *et al*., 2017; Di Piazza and Marraffa, 2002; Di Piazza and Musial, 2014; Fernandez *et al*., 2009; Gavriluț and Petcu, 2007; Gavriluț, 2008; Gavriluț, 2010; Gavriluț *et al*., 2015; Pap *et al*., 2018; Potyrala, 2007; Precupanu and Croitoru, 2002; Precupanu and Croitoru, 2003; Precupanu *et al*., 2010; Satco, 2007.

In this paper, we present some properties of the Birkhoff weak integral, defined in (Croitoru *et al*., 2017). Thus, a comparison result with Birkhoff simple integral and a characterization in terms of strongly Birkhoff weak integrability are established. The structure of this paper is the following: Section 1 is for Introduction. Some preliminary definitions and facts are recalled in Section 2. In Section 3, some properties of the Birkhoff weak integral are presented and Section 4 is for conclusions.

2. Preliminary Results

Suppose $(X, \|\cdot\|)$ is a Banach space, T is a nonempty set, $\mathcal{A}$ is a σ-algebra of subsets of T and $m:\mathcal{A}\to[0,\infty)$ is a non-negative set function with $m(\emptyset)=0$.

Definition 2.1 (Drewnowski, 1972) Let $m:\mathcal{A}\to[0,\infty)$ be a non-negative set function, with $m(\emptyset)=0$. The variation $\overline{m}$ of m is the set function $\overline{m}:\mathcal{P}(T)\to[0,\infty]$, defined by $\overline{m}(E)=\sup\left\{\sum_{i=1}^{n} m(A_i)\right\}$, for every $E\in\mathcal{P}(T)$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=1}^{n}\subset\mathcal{A}$, with $A_i\subseteq E$, for every $i\in\{1,\dots,n\}$.

Definition 2.2 (Drewnowski, 1972) A vector set function $\mu:\mathcal{A}\to X$ is called m-continuous (denoted $\mu \ll m$) if for every $\varepsilon>0$, there is $\delta>0$ such that for every $A\in\mathcal{A}$, with $\overline{m}(A)<\delta$, it results $\mu(A)<\varepsilon$.

Definition 2.3 (Croitoru *et al*., 2017) A vector function $f:T\to X$ is called *Birkhoff weak m-integrable* (*shortly Bw-m-integrable*) (*on T*) if there exists $\alpha\in X$ such that for every $\varepsilon>0$, there exist a countable partition P_ε of T and $n_\varepsilon\in\mathbb{N}$ so that for every other countable partition $P=\{A_n\}_{n\in\mathbb{N}}$ of T, with $P\geq P_\varepsilon$, and every $t_n\in A_n$, $n\in\mathbb{N}$ it holds $\|\sum_{k=0}^{n} f(t_k)m(A_k)-\alpha\|<\varepsilon, \forall n\geq n_\varepsilon$.

In this case, α is denoted by $(Bw)\int_T fdm$ and it is called *the Birkhoff weak integral of f* (*on T*) *relative to m*.

Remark 2.4 In the above definition, n_ε is uniform with respect to P and the choice of $t_n \in A_n$, $n \in \mathbb{N}$.

Example 2.5 I. Let $T = \mathbb{N}^* = \{1,2,\ldots\}$, $\mathcal{A} = \mathcal{P}(\mathbb{N}^*)$ and $m:\mathcal{A} \to [0,\infty)$ defined for every $A \in \mathcal{A}$ by $m(A) = 0$ if card $A < +\infty$ and $m(A) = 1$ if card $A = +\infty$. Then every real function $f:\mathbb{N}^* \to \mathbb{R}$, is Birkhoff weak m-integrable. Indeed, for every $\varepsilon > 0$, let $n_\varepsilon = 1$ and $P_\varepsilon = \left\{\{n\} \mid n \in \mathbb{N}^*\right\} \subset \mathcal{A}$, that is a countable partition of *T*. Consider $P = \{A_n\}_{n\in\mathbb{N}^*}$ an arbitrary countable partition of *T*, with $P \geq P_\varepsilon$ and every $t_n \in A_n$, $n \in \mathbb{N}^*$. It follows $A_n = \{n\}$ and $t_n = n$, for every $n \in \mathbb{N}^*$. So

$$\| \sum_{k=1}^{n} f(t_k) m(A_k) \| < \varepsilon, \forall n \geq n_\varepsilon,$$

wich shows that f is Birkhoff weak m-integrable and $(Bw)\int_T fdm = 0$.

II. Let $T = \left\{t_n \mid n \in \mathbb{N}^*\right\}$ be a countable set so that $\left\{t_n\right\} \in \mathcal{A}, \forall n \in \mathbb{N}^*$ and let $f:T \to X$ be a vector function having the property that the series $\sum_{n=1}^{\infty} f(t_n) m\left(\{t_n\}\right)$ is unconditionally convergent. Then *f* is Bw-m-integrable and $(Bw)\int_T fd\mu = \sum_{n=1}^{\infty} f(t_n) m\left(\{t_n\}\right)$.

Definition 2.6 (Candeloro *et al*., 2016) A vector function $f:T \to X$ is called *Birkhoff simple m-integrable* (*on T*) if there exists $\alpha \in X$ such that for every $\varepsilon > 0$, there exists a countable partition P_ε of T so that for every other countable partition $P = \{A_n\}_{n\in\mathbb{N}}$ of *T*, with $P \geq P_\varepsilon$ and every $t_n \in A_n, n \in \mathbb{N}$ it holds $\limsup_{n\to\infty} \| \sum_{k=0}^{n} f(t_k) m(A_k) - \alpha \| < \varepsilon$.

The vector α is denoted by $(Bs)\int_T fdm$ and it is called *the Birkhoff simple integral of* f *(on T) with respect to m*.

3. Some Results on Birkhoff Weak Integrability

In this section, some results on Birkhoff weak integrability are presented. We begin by establishing a comparative result with Birkhoff simple integral.

Theorem 3.1 *If* $f : T \to X$ *is Birkhoff weak m-integrable, then f is also Birkhoff simple m-integrable and the two integrals coincide.*

Proof. Let $\varepsilon > 0$ be arbitrary. Then there exist a countable partition P_ε of *T* and $n_\varepsilon \in \mathbb{N}$ satisfying the conditions of Definition 2.1. So, if we consider $P = \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ a countable partition of *T*, with $P \geq P_\varepsilon$ and $t_n \in A_n, n \in \mathbb{N}$, then

$$\| \sum_{k=0}^{n} f(t_k)m(A_k) - (Bw)\int_T fdm \| < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon.$$

This implies

$$\limsup_{n\to\infty} \| \sum_{k=0}^{n} f(t_k)m(A_k) - (Bw)\int_T fdm \| \leq \sup_{p\geq n_\varepsilon} \| \sum_{k=0}^{p} f(t_k)m(A_k) - (Bw)\int_T fdm \| \leq \frac{\varepsilon}{2} < \varepsilon,$$

that is, *f* is Birkhoff simple m-integrable and $(Bs)\int_T fdm = (Bw)\int_T fdm$.□

In the sequel, a characterization similar to the variational Henstock integrability is established for the Birkhoff weak integrability.

Definition 3.2 A vector function $f : T \to X$ is called strongly Bw-m-integrable if there is an m-continuous σ-additive measure $\mu : \mathcal{A} \to X$ having the property: for every $\varepsilon > 0$, there exist $n_\varepsilon \in \mathbb{N}$ and P_ε a countable partition of *T* such that for every other countable partition of*T*, $P = \{A_n\}_{n\in\mathbb{N}}$, $P \geq P_\varepsilon$ and every $t_n \in A_n, n \in \mathbb{N}$ it holds

$$\sum_{k=1}^{n} \| f(t_k)m(A_k) - \mu(A_k) \| < \frac{\varepsilon}{2}, \forall n \geq n_\varepsilon, \quad (1)$$

Theorem 3.3 *If* $f : T \to X$ *is strongly Bw-m-integrable, then f is also Bw-m-integrable.*

Proof. Consider $\alpha \in \mu(T)$. For every $\varepsilon > 0$ let $n'_\varepsilon \in \mathbb{N}$ and $P_\varepsilon = \{B_n\}_{n\in\mathbb{N}}$ given by Definition 2.3. Let $P = \{A_n\}_{n\in\mathbb{N}}$ be an arbitrary countable partition of *T*, with $P \geq P_\varepsilon$. Since $\alpha = \sum_{n=1}^{\infty} \mu(A_n)$, there exists $n''_\varepsilon \in \mathbb{N}$ such that $\| \sum_{k=1}^{n} \mu(A_k) - \alpha \| < \frac{\varepsilon}{2}, \forall n \geq n''_\varepsilon$. Let be $n_\varepsilon = \max\{n'_\varepsilon, n''_\varepsilon\}$, $n \geq n_\varepsilon$ and $t_n \in A_n, n \in \mathbb{N}$.

Then, by (1) it results

$$\| \sum_{k=1}^{n} f(t_k)m(A_k) - \alpha \| \leq \| \sum_{k=1}^{n} f(t_k)m(A_k) - \sum_{k=1}^{n} \mu(A_k) \| + \| \sum_{k=1}^{n} \mu(A_k) - \alpha \| <$$

$\sum_{k=1}^{n} \| f(t_k)m(A_k) - \mu(A_k) \| + \frac{\varepsilon}{2} < \varepsilon$. Consequently, *f* is Bw-m-integrable. □

If *m* is σ-additive, then the converse of Theorem 3.3 is also valid.

Theorem 3.4 *Suppose m is σ-additive. If* $f: T \to X$ *is Bw-m-integrable on every set* $A \in \mathcal{A}$*, then f is strongly Bw-m-integrable.*

Proof. Let $\mu: \mathcal{A} \to X$ be defined by $\mu(A) = (Bw)\int_A f d\mu, \forall A \in \mathcal{A}.$ Considering $F = \{f\}$, from Theorem 22 (Croitoru *et al*., 2015) it results that μ « *m.* And according to Theorem 23 (Croitoru *et al*., 2015), it follows that μ is *σ*-additive. Now, (1) holds by Definition 2.1 and the theorem is proved. □

4. Conclusion

In this paper, we present some properties of Birkhoff weak integral for vector functions with respect to a non-negative set function. So, a comparison result with Birkhoff simple integral and a characterization in terms of strongly Birkhoff weak integrability are established.

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REZULTATE ASUPRA INTEGRABILITĂȚII SLABE BIRKHOFF

(Rezumat)

Prezentăm unele proprietăți ale funcțiilor vectoriale m-integrabile slab Birkhoff în raport cu o funcție de mulțime nenegativă m. S-a stabilit un rezultat de comparație cu integrala Birkhoff simplă și o caracterizare în termeni de integrabilitate slabă Birkhoff tare.